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Groebner Basis and its Applications

Alok Kumar	Assistant Professor, Deshbandhu College, Delhi University. New Delhi-110019.
Rahul Singh	Assistant Professor, Ramanujan College, Delhi University. New Delhi-110019.
Dheeraj Tiger	Assistant Professor, Rajdhani College, Delhi University. New Delhi-110015
Sanjay Goplani	Associate Technical Architect, Tech Mahindra, Oberoi Garden Estate, Candivali, Andheri (East), Mumbai-400072.

ABSTRACT

Groebner Bases is a technique that provides algorithmic solutions to a variety of problems in Commutative Algebra and Algebraic Geometry. Bruno Buchberger algorthim for computing Groebner bases is a powerful tool for solving many important problem in Commutative Algebra and Algebraic Geometry. The theory of Groebner bases is centered around the concept of ideals generated by finite sets of multivariate polynomials. In the present paper Groenber Basis is used to solve the system of multivariate polynomial and integer programming problems.

KEYWORDS multivariate polynomial, Groebner basis, S-polynomials, Affine space

1. Introduction

As algebra becomes more widely used in a variety of applications and computers are developed to allow efficient calculations in the field so there becomes a need for new techniques to further this area of research problems. The theory of Groebner bases is centered around the concept of ideals generated by finite sets of multivariate polynomials. Therefore, we start our discussion by defining some basic algebraic structures, and move on to the notion of ideals.

A commutative ring $(R, +, .)$ is a set R with the two binary operations addition (+) and multiplication (.) defined on R such that:

1) $(R, +, .)$ is a commutative group,
2) . is commutative and associative
3) distributive law $a.(b+c) = a.b + a.c$ holds $\forall\ a, b, c \in R$

A commutative ring with a multiplicative identity $(R, +, .)$ is called a field if every nonzero element of R has a multiplicative inverse in R.

Monomial

Let N denote the non-negative integers. Let $\alpha = (\alpha_1, \alpha_2, \ldots \ldots \ldots \alpha_n)$ be a power vector in

N^m, and let $x_1, x_2, \ldots \ldots \ldots x_n$ be any n variables. Then a monomial x^α in $x_1, x_2 \ldots \ldots \ldots x_n$ is defined as the product $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots \ldots \ldots x_n^{\alpha_n}$

A monomial in $x_1, x_2, \ldots \ldots \ldots x_n$ is a product of the form $x_1^{\alpha_1} x_2^{\alpha_2} \ldots \ldots \ldots x_n^{\alpha_n}$ where all of the exponents $\alpha_1, \alpha_2, \ldots \ldots \ldots \alpha_n$ are nonnegative integers. The total degree of this monomial is the sum $\alpha_1 + \alpha_2 + \cdots \ldots \ldots \alpha_n$

A multivariate polynomial f in $x_1, x_2 \ldots \ldots \ldots x_n$ with coefficients in a field k is a finite linear combination, $f(x_1, x_2, \ldots \ldots \ldots x_n) = \sum_\alpha a_\alpha x^\alpha$ of monomials x^α and coefficients $a_\alpha \in K$. The set of all polynomials in $x_1, x_2 \ldots \ldots \ldots x_n$ with coefficients K is denoted $k[x_1, x_2, \ldots \ldots \ldots x_n]$ Let $f = \sum_\alpha a_\alpha x^\alpha$ be a polynomial in $k[x_1, x_2, \ldots \ldots \ldots x_n]$ then a_α be the coefficients of the monomial x^α, if $a_\alpha \neq 0$ then we call $a_\alpha x^\alpha$ a term of f.

1.1Affine Space $k^n = \{(a_1, a_2, a_3 \ldots \ldots \ldots a_n) : a_1, a_2, a_3 \ldots \ldots \ldots a_n \in k\}$ is an affine space where k is a field and $n \in Z$.

Let $k[x_1, x_2, \ldots \ldots \ldots x_n]$ denote the set of all polynomials in n variables with coefficients in the field k and $k[x_1, x_2, \ldots \ldots \ldots x_n]$ is a

commutative ring with respect to operations addition and multiplication of polynomials. Let $f \in k[x_1, x_2, \dots\dots\dots\dots x_n]$ Define $V(f)$ to be the set of solutions of the equation $f = 0$. i.e. $V(f) = \{(a_1, a_2, a_3 \dots\dots a_n) \in k^n : f(a_1, a_2, a_3 \dots\dots\dots a_n) = 0\} \subset k^n$
The $V(f)$ is called the affine variety.Similarly if $f_1, f_2, f_3\_\_\_\_ f_s \in k[x_1, x_2, \dots\dots\dots\dots x_n]$ the variety $V(f_1, f_2, f_3\_\_\_\_ f_s)$ is defined to be the set of all solutions of the system

$$f_1 = 0, f_2 = 0, f_3 = 0 \dots\dots\dots\dots, f_s = 0$$

i.e.

$$V(f_1, f_2, f_3\_\_\_\_ f_s) = \{(a_1, a_2, a_3 \dots\dots a_n) \in k^n : f_i(a_1, a_2, a_3 \dots\dots a_n) = 0 \text{ for all } 1 \le i \le s$$

2. Polynomial reduction and S-Polynomial of two Polynomial

Suppose, $g, h \in k[x_1, x_2, \dots\dots x_n]$, with $f \neq 0$,we say that g reduces to h modulo f, written as

$$f: g \to h$$

if and only if $lp(g)$ divides a nonzero term X that appears in f and

$$h = g - \frac{X}{lt(f)} f.$$

Similarly a polynomial g reduces to another polynomial h modulo some polynomial set F denoted as

$$F: g \to h$$

if and only if the LT(g) can be deleted by the subtraction of an appropriate polynomial f in F a monomial u where $u = \frac{LM(g)}{LM(f)}$, and a scalar b in k , where $b = \frac{LM(g)}{LM(f)}$, and a scalar b in k, where $b = \frac{LC(g)}{LC(f)}$, yielding h .

2.1 S-Polynomial

Given two polynomials $, g \in k[x_1, x_2, \dots\dots\dots x_n]$ let $J = l.c.m(LM(f), LM(g))$ We define the S-polynomial of f and g as the linear combination

$$S - poly(f, g) = \frac{J}{LT(f)} . f - \frac{J}{LT(g)} . g$$

Since $\frac{J}{LT(f)} . f$ and $\frac{J}{LT(g)} . g$ are monomials, then the $S - poly(f, g)$ is a linear combination with polynomial coefficients of f and g , and belongs to the same ideal generated by f and g .

3. Groebner Basis and it's applications

A set of non-zero polynomials $G = \{g_1, g_2 \dots g_n\}$ contained in an ideal I, is called a Groebner basis for I, if and only if for all $f \in I$ such that $f \neq 0$, there exists $i \in 1 \dots\dots\dots t$ such that $l_p(g_i)$ divides $l_p(f)$. Let $G = \{g_1, g_2, \dots\dots\dots g_n\}$ be a finite set of polynomials. Let I be the ideal generated by G. The following are equivalent:

(a) G is a Groebner basis.

(b) $\forall\ g_i, g_j \in G: S - poly(g_i, g_j) = 0$ reduces to zero modulo G.

(c) Every reduction of an g of I to a reduced polynomial with respect to G always gives zero. The following algorithm is given by Buchberger to compute groebner basis.

Input: A polynomial set $F = \{f_1, f_2, \dots f_n\}$ that generates an ideal I.

Output: A Groebner basis $G = \{g_1, g_2, \dots g_n\}$ that generates the same ideal I. with $F \subset G$

Step

$$G = F$$
$$M = \{\{f_i, f_j\}: f_i, f_j \in G \text{ and } f_i \neq f_j\}$$

Repeat

$$\{p, q\} = a \text{ pair in } M$$
$$M = M - \{p, q\}$$
$$S = Spoly(p, q)$$
$$h = normal\ form(S, G)$$
$$if\ h \neq 0\ then$$
$$M = M \cup \{\{g, h\}\ \forall\ g \in G\}$$
$$G = G \cup \{h\}$$
$$Until\ M = \emptyset,$$

2.2 Algorithm Generalized Division

put: A polynomial set $F = (f_1, f_2, - - - f_s)$ d any nonzero polynomial f in $[x_1, x_2, \ldots \ldots \ldots x_n]$.
utput: The remainder r, of dividing f by F. The uotients $q_1, q_2, \ldots \ldots q_s$ such that $f = q_1 f_1 + f_2 + \cdots \ldots \ldots \ldots q_s f_s + r$ with either $r =$ or r is a completely reduced polynomial with spect to F.

$$q_i = 0 \; for \; i = 1, \ldots \ldots s$$
$$r = 0$$
$$p = f$$

Repeat

$$i = 1$$

dividing = true
While ($i \leq s$) and (dividing) do
If $LT(f_i)$ divides $LT(p)$ then

$$u = \frac{LT(p)}{LT(f_i)}$$
$$q_i = q_i + u$$
$$p = p - u.f_i$$

dividing = false
else $i = i + 1$
If not dividing then

$$r = r + LT(p)$$
$$p = p - LT(p)$$

Until $p = 0$

Theorem: The general polynomial division algorithm ends in a finite number of steps.

Proof : Fix a monomial ordering. Using the algorithm above to divide a polynomial $f \in R$ by an ordered set $G = \{g_1, g_2, \ldots \ldots \ldots g_m\}$ initially set the remainder r and the quotients $q_1, q_2, - - - q_m$ to zero. If the leading term of f is not divisible by any of $\{LT(g_i) \ldots \ldots \ldots LT(g_m)\}$ then $LT(f)$ is added to the remainder and f is replaced by $f - LT(f)$. Note that the leading term of $f - LT(f)$ is strictly less than (f) . Otherwise, at some step in the algorithm (f) . is divisible by $LT(g_i)$, for some i. Then the leading monomial of the polynomial $f - \frac{LT(f)}{LT(g_i)} g_i$ is strictly less than the leading monomial of f. Under any monomial ordering, there is a finite number of monomials strictly less than a given monomial. That is, there is no infinite strictly decreasing sequence of monomials in R under any ordering. Since the leading monomial of f is strictly decreasing at each iteration of the algorithm, it must end in a finite number of steps.

A Groebner basis $G = \{g_1, g_2, \ldots \ldots g_t\}$ is called reduced if for all i, $lc(g_i) = 1$, and for all $i \; of \; j$. $lp(g_i)$ *does not divide* $lc(g_j)$.

Suppose $G = \{g_1, g_2, \ldots \ldots \ldots g_t\}$ be a Groebner basis for the ideal I. If $lp(g_2)$ divides (g_1) , then $\{g_2, \ldots \ldots \ldots g_t\}$ is also a Groebner basis for I .

Computation of reduced groebner basis from groebner basis

Let $G = \{g_1, g_2, \ldots \ldots g_t\}$ be a Groebner basis for the ideal I. To obtain a reduced Groebner basis from G, eliminate all g_i for which there exists $j \; of \; i$ such that $lp(g_j)$ divides $lp(g_i)$ and divide each remaining g_i by $lc(g_i)$.

How can Grobener Basis be Applied ?

The general strategy: Given a set F of polynomials in $k[x_1, x_2, \ldots \ldots \ldots \ldots x_n]$ (that describes some problem, e.g. a system of equations)

1. We transform F into another set G of polynomials "with certain nice properties" (called a "Groebner basis") such that F and G are "equivalent" (i.e. generate the same ideal).
2. From the theory of Groebner Basis we know that this transformation can be carried out by an algorithm.
3. From the theory of Groebner Basis we know that, because of the "nice properties of Groebner Basis many problems that are difficult for general F are "easy" for Groebner bases G.
4. We solve the problem for G and transform the solution back to F.

3. Properties of Groebner basis

Groebner bases yields some very useful algebraic results. Now we have been discussing what is known as the Ideal

Membership Problem. The remainder of a polynomial in $k[x_1, x_2, \dots \dots \dots - - x_n]$ on division by a Groebner basis is unique. If $G = \{g_1, g_2, - - - g_t\}$ is a Groebner basis for I *and* $f \in k[x_1, x_2, - - \dots \dots x_n]$ then $f \in I$ if and only if the remainder of f on division by $\{g_1, g_2, \dots \dots \dots g_t\}$ is zero. If $G = \{g_1, g_2, - - - g_t\}$ is a Groebner basis for I *and* $f \in k[x_1, x_2, - - \dots \dots x_n]$ then f can be written uniquely in the form $f = g + r$ where $g \in I$ and no term of r is divisible by any $LT(g_i)$.

3.1 Elimination ideal :

Given $I =< f_1, f_2, \dots \dots \dots \dots f_s > \subset k[x_1, x_2, \dots - - - - - x_n]$ the lth, elimination ideal I_l is the ideal of $k[x_{l+1}, x_2, - - - - - - x_n]$ defined by

$$I_l = I \cap k[x_{l+1}, x_2, \dots \dots \dots \dots \dots x_n]$$

So the elements of I_l are all the equations that follow from $f_1 = = f_s == 0$ and eliminate the variables $x_1, x_2, \dots \dots \dots \dots x_l$. We can easily say that I_l is indeed an ideal of $k[x_{l+1}, x_2, - - - - - - x_n]$.

Theorem: The lth, elimination ideal I_l, Defined above, is an ideal of $k[x_{l+1}, x_2, \dots \dots \dots - - x_n]$.

Proof: Let I be an ideal and let $I_l = I \cap k[x_{l+1}, x_2, - - - - - - x_n]$. Then $0 \in I_l$ because $0 \in I$ and $0 \in k[x_{l+1}, x_2, - - - - - - x_n]$. I_l is closed under addition, since I and $k[x_{l+1}, x_2, - - - - - - x_n]$ closed under addition, and the sum of any two polynomials with variables $x_{l+1}, x_2, - - - - - - x_n$ must be a polynomial with these variables. It only remains to show that I l absorbs elements of the ring $k[x_{l+1}, x_2, \dots \dots \dots \dots x_n]$. Under multiplication. Pick $f \in I_l$ and $g \in k[x_{l+1}, x_2, - - - - - - x_n]$. Then $f \in I$ and so $fg \in I$. also $\in k[x_{l+1}, x_2, \dots \dots \dots \dots \dots x_n]$ and so $fg \in I_l$. Therefore I_l is an ideal.

Theorem: If $I =< f_1, f_2, \dots \dots \dots \dots f_s > \subset k[x_1, x_2, \dots - - - - - x_n]$ is an ideal and $G = \{g_1, g_2, - - - g_t\}$ is a Groebner basis for I for lex order with $x_1 > x_2 > \cdots - - - > x_n$, then for each $0 \le l \le n$, the set $G_l = G \cap k[x_{l+1}, x_{l+2}, \dots \dots \dots \dots \dots x_n]$ is a Groebner basis for the elimination ideal. $I_l = I \cap k[x_{l+1}, x_{l+2}, \dots \dots \dots \dots \dots x_n]$

Proof : Since $G \subset I$, we know $G_l \subset I_l$. Let $f \in I_l$. We need to show that $LT(f)$ is divisible by $LT(g)$ for some $\in G_l$. By denition of I_l, we know that $f \in I$ and thus $LT(f)$ is divisible by $LT(g)$ for some $g \in G$, since G is a Groebner basis of I. Since $f \in I_l$, its leading term $LT(f)$ involves only the variables $x_{l+1}, x_{l+2}, - - - - - - x_n$. So the same must be true for (g) . Note that since we are using lex ordering with $x_1 > x_2 > \cdots \dots \dots > x_n$, any monomial involving $x_1, x_2, - - - - \dots x_l$ is greater than all monomials in $k[x_{l+1}, x_{l+2}, \dots \dots \dots \dots \dots x_n]$. It follows that $(g) \in k[x_{l+1}, x_{l+2}, \dots \dots \dots \dots \dots x_n]$ implies that $g \in k[x_{l+1}, x_{l+2}, \dots \dots \dots \dots \dots x_n]$. Thus $\in G \cap k[x_{l+1}, x_{l+2}, \dots \dots \dots \dots \dots x_n] = G_l$. So we have shown that any $\in I_l = I \cap k[x_{l+1}, x_{l+2}, \dots \dots \dots \dots \dots x_n]$ is divisible by $LT(g)$ for some $g \in G_l = G \cap k[x_{l+1}, x_{l+2}, \dots \dots \dots \dots \dots x_n]$. Therefore, G_l is a Groebner basis of I_l by definition of Groebner basis.

Groebner basis algorithm has been intensively studied and more applications have been exploited. One of the most important applications is the use of Groebner bases algorithm for solving systems of polynomial equations and answering questions about the solvability of such systems.

We assume that the systems of equations we are dealing with are in the variables $x_1, x_2, - - - \dots, x_n$ with the lexicographic order $x_1 > x_2 > \cdots - - - > x_n$.

Given $I =< f_1, f_2, \dots \dots \dots \dots f_s > \subset k[x_1, x_2, - - - - - - x_n]$ defined by:

$I_l = I \cap k[x_{l+1}, x_{l+2}, \dots \dots \dots \dots \dots x_n]$

Thus, I_l consists of all consequences of $f_1 = f_2 === \cdots \dots \dots \dots f_s = 0$ which eliminate the variables $x_1, x_2, - - - - - - x_l$.

We see that eliminating $x_1, x_2, \dots \dots - - - - x_l$ means finding nonzero polynomials in the lth elimination ideal I_l, and the significance of Groebner bases for solving systems of equations from the fact that, for Groebner bases, it is simple to construct all of the elimination ideals.

Consider the following system of equations:

$$x^2 + y^2 + z^2 = 1$$
$$x^2 + y^2 + z^2 = 2x$$
$$2x + 3y + z = 0$$

Let I be the ideal $I = (x^2 + y^2 + z^2 - 1, x^2 + y^2 + z^2 - 2x, 2x - 3y - z)$ then a Groebner basis for I with respect to the lex order is $G = (g_1, g_2, g_3)$ where

$$g_1 = 2x - 1$$
$$g_2 = 3y + z - 1$$
$$g_3 = 40z_2 - 8z - 23$$

There is exactly one generator in the variables $x_{n-1} = y$ and $x_n = z$. Since we have all possible roots of z, we can determine the roots of y. It is possible that we have a generator in x_{n-1} alone. Generator g_1 is in x alone. All roots of x can be computed. The process of back substitution continues until all roots of generators are determined.

A system, F, has finitely many solutions if and only if for all $i(1 \le i \le n)$ a power product of the form x_i^h occurs among the leading power products of the polynomials in Groebner basis of F, where n is the number of polynomials in F. A system F is unsolvable if and only if $1 \in G$ where G is the Groebner basis generated by the set of polynomials in the system.

Let f be any polynomial s.t. $f \in k[x_1, x_2, \dots - - - - - x_n]$ we define $V(f)$ to be the set of solutions of the equation $f = 0$. $V(f) = \{(a_1, a_2 \_\_\_\_ a_n) \in k_n : f(a_1, a_2 \_\_\_\_ a_n)\} = 0 \subset k_n$ is called the variety defined by f. More generally, given $f_1, f_2, - - - - f_s \in k[x_1, x_2, \dots \dots \dots \dots x_n]$ the variety $V(f_1, f_2, \dots - - - f_s)$ is defined to be the set of all solutions of the system $f_1 = f_2 = \cdots \dots \dots f_s = 0$.

If $S \subset k[x_1, x_2, \dots \dots \dots \dots x_n]$ we define $V(f) = \{(a_1, a_2 \_\_\_\_ a_n) \in k_n : f(a_1, a_2 \_\_\_\_ a_n) = 0 \text{ for all } f \in S\}$

Now we want to illustrate how one can apply the technique of Groebner bases to determine whether a given graph can be 3-colored. We are given a graph G with n vertices with at most one edge between any two vertices. We want to color the vertices in such way that only 3 colors are used, and no two vertices connected by an edge are colored the same way. If G can be colored in this fashion, then G is called 3-colorable. First, we let $\xi = e^{(\frac{2\pi i}{3})} \in C$ be a cube root of unit $(\xi^3 = 1)$ We represent the 3-colors by $1, \xi, \xi^2$, the 3 distinct cube foots of unity. Now, we let $x_1 - \dots \dots \dots x_n$ be variables representing the distinct vertices of the graph . Each vertex is to be assigned one of the 3 colors $1, \xi, \xi^2$. This can be represented by the following n equations:

$$x_i^3 - 1 = 0, 1 \le i \le n$$

also if the vertices x_i *and* x_j are connected by an edge, they need to have a different color. Since $x_i^3 = x_j^3$, we have $(x_i - x_j)(x_i^2 + x_i)(x_j^2 + x_j) = 0$, Therefore x_i *and* x_j will have different colors if and only if

$$x_i^2 + x_i x_j + x_j^2 = 0$$

The graph G is 3-colorable if and only if $V(I) \neq \phi$

Theorem: A graph is 3-colorable if and only if the set of polynomials associated with our graph have a common solution in the complex numbers.

Proof: Suppose the given graph is 3-colorable. Recall that $x_i^3 - 1 = 0$ for each i. Then for adjacent vertices x_i, x_j we have $x_i^3 = x_j^3 = (x_i - x_j)(x_i^2 + x_i)(x_j^2 + x_j) = 0$. The adjacent vertices are colored differently, so $(x_i - x_j) \neq 0$. Then $(x_i^2 + x_i)(x_j^2 + x_j) = 0$ for some x_i *and* x_j . This is true for any pair of adjacent vertices. Then the set of polynomials associated with the vertices have at least one common solution. Now suppose that the polynomials associated with adjacent vertices have a common solution. This means that there exists x_i, x_j such that $(x_i^2+x_i)(x_j^2+x_j) = 0$ for all pairs of adjacent vertices. Notice that $(x_i \neq x_j)$ for this to be true. Then $x_i^3 - x_j^3 = 0$ and we know from above that x_i *and* x_j will be assigned different colors. Hence the graph is 3-colorable.

5. Integer Programming

Now we introduce a new approach for solving integer programming problems (IP_s) with the help of Groebner basis. Given an $m \times n$ matrix A, and an m-vector b, the 0-1 feasibility integer programming problem is the problem of deciding whether there is an n-vector x with 0-1 coordinates such that

$$Ax = B, x \in \{0,1\}$$

This problem can be rewritten equivalently as the following system of equations,

$$Ax = B,$$

$$x_j^2 = x_j, j = 1,2,3 \dots \dots \dots \dots n$$

From Hilbert Basis Theorem for every polynomial ideal and every term order, there is a Groebner basis that has a finite number of elements. Given an ideal F generated by polynomials $f_1, f_2, \dots, f_s$ and a term order, we can compute the Groebner basis of F using Buchbergers algorithm. We use $I(V)$ to denote the ideal that contains all polynomials that vanish on a given variety V and $V(I)$ to denote the variety $V(r_1, r_2, \dots r_s)$ where $G = (r_1, r_2, \dots r_s)$ is a Groebner basis of the ideal I.

Given $I = < f_1, f_2, \dots \dots f_s > \subset k[x_1, x_2, \dots \dots x_n]$ the lth elimination ideal I_l is defined as

$I_l = I \cap k[x_{l+1}, x_2, \dots \dots x_n]$

The Elimination Theorem:

Let $I \subset k[x_1, x_2, \dots \dots x_n]$ be an ideal and let G be a Groebner basis of I Then for every $0 \le l \le n$ the set $G_l = G \cap k[x_{l+1}, x_{l+2}, \dots \dots x_n]$ is a Groebner basis of the lth elimination ideal I_l.

Proof: An ideal I is radical if $f^m \in I$ for any integer $m \ge 1$ implies that $f \in I$. Let $I \subset k[x_1, x_2, \dots \dots x_n]$ be an ideal. The radical of I denoted $\sqrt{I}$, is the set

$$\{f : f^m \in I \text{ for some integer } m \ge 1\}$$

Let k be an algebraically closed field. If I is an ideal in $k[x_1, x_2, \dots \dots x_n]$, then $I(V(I)) = \sqrt{I}$

Let k be an algebraically closed field *If* $f_1, f_2, \dots f_s \in k[x_1, x_2, \dots \dots x_n]$ are such that $f \in I(V(f_1, f_2, \dots f_s))$, then there exists an integer $m \ge 1$ such that $f^m \in < f_1, f_2, \dots f_s >$ and conversely.

$\{0 - 1\}$ Integer Programming

Consider the following polynomial in $k[x_1, x_2, \dots \dots x_n]$

$$f_i = \sum_{j=1}^{n} a_{ij}x_j - b_i, i = 1,2, \dots \dots m.$$

$$g_j = x_j^2 - x_j, j = 1,2, \dots \dots n.$$

and let

$W = V(f_1, f_2 \_\_\_ f_m, g_1, g_2 \_\_\_ g_n)$ be the variety and J be the ideal defined by:

$$J = (f_1, f_2 \_\_\_ f_m, g_1, g_2 \_\_\_ g_n)$$

$V(G_1)$.

Consider the Integer programming

$$x_1 + x + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 = 15, x_j \in \{0,1\} \text{ for all } j$$

The ideal corresponding to above IP is:

$$J = < x_1 + x + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 - 15, x_1^2 - x_1, x_2^2 - x_2, x_3^2 - x_3, x_4^2 - x_4, x_5^2 - x_5, x_6^2 - x_6 >$$

The Groebner basis of J is:

$$G = \{x_6^2 - x_6, x_5 + x_6 - 1, x_4 + x_6 - 1, x_3 + x_6 - 1, x_2 + x_6 - 1, x_1 + x_6 - 1\}$$

Now compute $G_5 = \{x_6^2 - x_6\}$ indicating $a_6 = 1$ *or* $a_6 = 0$. Similarly compute

G_4, G_3, G_2, G_1, and with the help of this find a_1, a_2, a_3, a_4, a_5.

We find two feasible point of integer programming: (1,1,1,1,1,0) and (0,0,0,0,0,1).

Consider the optimization problem:

minimize cx

subject to $Ax = b$

$$x_j^2 = x_j \text{ for all } j$$

Now we work in $k[x_1, x_2, ------ x_n, y]$ and let

$h = y - \sum_{j=1}^{n} c_j x_j$ $W = V(f_1, f_2, \_\_\_\_\_\_ g_1, g_2, \dots\dots\dots\dots\dots\dots h)$

and the ideal J is defined as:

$J =< f_1, f_2, \_\_\_\_\_\_ g_1, g_2, \dots\dots ------- . h >$

Now to optimize the 0-1 integer programming we apply the same algorithm which we discuss above.

The following example illustrates IP with unequal constraints .

Consider the IP

$$x_1 + x_2 + x_3 \le 2$$
$$x_2 + x_3 + x_4 \le 4$$
$$x_i \in \{0,1\}, i = 1,2, \dots 4$$

We introduce slack variables s_1 and s_2 for the first and second constraint, respectively. Clearly, $s_1 \le 2$ and $s_2 \le 4$. Furthermore, we rewrite these variables as $s_1 = x_5 + 2x_6$ and $s_2 = x_7 + 2x_8 + 4x_9$ with $x_5 \dots\dots\dots\dots x_9 \in \{0,1\}$

so we can rewrite the IP as:

$$x_1 + x_2 + x_3 + x_5 + 2x_6 = 2$$
$$x_2 + x_3 + x_4 + x_7 + 2x_8 + 4x_9 = 4$$
$$x_i^2 - x_i = 0, i = 1,2, ------- 9$$

Now we can solve this problem by previous algorithm because now this problem change into equality and {0,1} IP problem.

6. Results and Discussion

First we show result for finding Groebner basis

Input: Let F be set of multivariate polynomial f_1, f_2, f_3 where

$$f_1 = xy - 2yz - z$$
$$f_2 = y^2 - x^2z + xz$$
$$f_3 = z^2 - y^2x + x$$

Output: the Groebner basis for the multivariate polynomial set F is: $G = x^4 + z^3 - 17x^4 + 3z^5 + z^6 + 60x^7 - 29z^8 + 124z^9 - 48z^{10} + 64z^{11} + 64z^{12}, -22001x + 14361yz + 16681z^2 + 26380z^3 + 226657z^4 + 11085x^5 - 90346z^6 - 472018z^7 - 520424z^8 - 139296z^9 - 150784x^{10} + 490368z^{11}, 43083y^2 - 11821z + 267025z^2 - 583085z^3 + 663460z^4 - 2288350x^5 + 2466820z^6 - 3008257x^7 + 4611948x^8 - 2592304z^9 + 2672704x^{10} - 1686848z^{11}, 43083x - 118717z + 69484x^2 + 402334x^3 + 409939z^4 + 1202033x^5 - 2475608z^6 + 354746z^7 - 6049080x^8 + 2269472z^9 - 3106688x^{10} + 3442816z^{11}$ $t^2 + x^2 + y^2 + z^2 = 0$

$$t^2 + 2x^2 - xy - z^2 = 0$$

$t + y^3 - z^3 = 0$

and I be the ideal generated by these polynomial

OUTPUT Groebner basis for I with respect to lex order is

$$G = \{g_1, g_2, g_3, g_4, g_5\}$$
$$g_1 = x^2 + y^2 + z^2 + y^6 - 2y^3z^3 + z^6$$
$$g_2 = 2y^2 + 3x^2 = y^6 - 2y^3z^3 + z^6 + xy$$
$$g_3 = -5y_7^3yz_5^2y^7 + 10y^4z^3 - 3yz^6 + 6x^5y^2 + 4y^8z^2 - 5y^5z^6 + 2z^9y^2 - 3y^5z^2 - y^{11} + 3xz^2 + xz^6$$
$$g_4 = t + y^3 - z^3$$
$$g_5 = 13y^2z^2 + 9z^4 + 6y^6z^2 - 12z^5y^3 + 6z^8 + 5z^6y^2 + 6z^6y^6 - 4z^9y^3 + z^{12} + 5y^8 - 10y^5z^3 - 4y^9z^3 + y^{12}$$

The polynomial corresponding to G are:

$x_i^2 - 1$ *for* $i = 1,2, \dots\dots\dots 8$ *and* $x_i^2 + x_ix_j + x_j^2$ *for the pairs* $(i, j) \in (1,2), (1,5), (1,6), (2,3), (2,4), (2,8), (3,4), (3,8), (4,5)/(5,6), (5,7), (6,7), (7,8)$

Now Groebner basis G for the ideal I corresponding the above polynomial .

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$$G = \{x_1 - x_7, x_2 + x_7 + x_8, x_3 - x_7, x_4 - x_8, x_5 + x_7 + x_8, x_6 - x_8, x_7^2 + x_7x_8 + x_8^2, x_8^2 - 1\}$$

since 1 does not belong to G so $V(I) \neq \emptyset$,and hence graph is 3- colorable.

The polynomial corresponding to G are:

$x_i^2 - 1 \; for \; i = 1,2, \_\_\_ .9 \; and \; x_i^2 + x_ix_j + x_j^2 \; for \; the \; pairs \; (i,j) \in (1,2), (1,4), (1,5), (1,6), (2,3), (2,5)(2,7), (3,4)(3,6), (3,9), (5,6), (6,8), (7,8), (8,9)$

Now Groebner basis G for the ideal I corresponding the above polynomial .

$$G = \{x_1^2 - 1, x_8^2 + x_8x_9 + x_9^2, x_3^2 + x_9x_3 + x_9^2, (x_7 - x_9)(x_7 + x_8 + x_9), x_1 + x_7 + x_8, (-x_5 + x_7)(x_8 - x_5), (x_5 - x_9)(x_4x_7 + x_4x_8 + x_7x_8 + x_4x_9 + x_7x_9 + x_8x_9), (x_4 - x_5)(x_4 + x_7 + x_8), x_3x_7 + x_3x_8 - x_7x_8 + x_3x_9 + x_9^2, x_3x_4 + x_4x_5 - x_4x_7 + x_5x_7 - x_4x_8 + x_5x_8 - x_3x_9 - x_9^2, x_2 + x_7 + x_8, x_1 + x_5 - x_7 - x_8\}$$

since 1 does not belong to G so $V(I) \neq \emptyset$,and hence graph is 3- colorable.

Optimize 0-1 integer programming Consider the IP

Minimize: $x_1 + 2x_2 + 3x_3$

subject to: $x_1 + 2x_2 + 3x_3 = 3$

$x_j^2 = x_j \; for \; all \; j$

Now $J =< y - x_1 - 2x_2 - 3x_3, 2x_1 + 4x_2 + 6x_3 - 6, x_1^2 - x_1, x_2^2 - x_2, x_3^2 - x_3 >$

and Groebner basis is :

$$G = \left\{ \begin{matrix} 12 - 7y + y^2, 3 + x_3 - y, y + x - 4, \\ 1 - x_1 \end{matrix} \right\}$$

Now the root of the polynomial $12 - 7y + y^2$ are 3 and 4

Minimum value of y is 3, so optimal value is 3 and the corresponding solution is (1,1,0).

Solution of 0-1 integer programming Consider the Integer programming :

$$x_1 + x + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 = 15, x_j \in \{0,1\} \; for \; all \; j$$

The ideal corresponding to above IP is:

$$J =< x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 - 15, x_1^2 - x_1, x_2^2 - x_2, x_3^2 - x_3, x_4^2 - x_4, x_5^2 - x_5, x_6^2 - x_6 >$$

The Groebner basis of J is:

$$G = \{x_6^2 - x_6, x_5 + x_6 - 1, x_4 + x_6 - 1, x_3 + x_6 - 1, x_2 + x_6 - 1, x_1 + x_6 - 1\}$$

Now compute $G_5 = \{x_6^2 - x_6\}$ indicating $a_6 = 1 \; or \; a_6 = 0$. Similarly compute G_4, G_3, G_2, G_1. and with the help of this find a_1, a_2, a_3, a_4, a_5.

We find two feasible point of integer programming: (1,1,1,1,1,0) and (0,0,0,0,0,1).

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