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CORRESPONDENCE OF IDEALS IN GROUP RINGS

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ABSTRACT

Let G be any group and R and S be rings. If the group rings RG and SG are isomorphic as rings, a correspondence between the ideals of R and S has already been established in [4]. Here we study under what conditions, the correspondence obtained in [4] is one-to-one.

1. Introduction :

The second isomorphism problem asks whether or not R is an invariant of RG i.e. if RG≅SG as rings, whether R is isomorphic to S or not. The answer is no in general. In [4] we took up the case when RGSG and formulated a correspondence between the ideals of R and S. we studied some properties of this correspondence and proved that it worked for Jordan ideals [1] and Lie ideals [1] provided that G is an abelian group. Here we prove that this correspondence is one-to-one in case the second isomorphism problem is resolved.

Proposition 2.1

Let R and S be rings, G any group and $\theta: RG \cong SG$ a ring isomorphism. For every ideals I of R, set

$$\phi(I)=\{s\in S \mid s1_G \in \theta(IG)\}$$

(i) Then $\phi(I)$ is an ideal of S and for I=R, $\phi(I)$ =S. Thus we have a correspondence.

$$\theta\, L_I(R)\rightarrow L_I(S)$$

Where $L_I(R)$ and $L_I(S'$ denote lattices of ideals of R and S respectively.

Similarly we have a correspondence

$$\psi : L_I(R)\rightarrow L_I(S)$$

(ii) If θ induces an isomorphism between R and S we have $\phi\,\psi$ =1, $\psi\,\phi$ =1

i.e. ϕ is a one-to-one correspondence.

Proof :

(i) has already been proved in [4]

(ii) since ϕ (I) is an ideal of S,

$\therefore$ ϕ(I) G is an ideal of SG.

Take any element

$$\sum s_g g \in\phi(I)G, s_g \in\phi(I)\ \forall g\in g$$

Then

$$\theta^{-1}\left(\sum s_g g\right)=\theta^{-1}\left(\sum (s_g 1_G)\,(1_S g)\right)$$

$$=$$

$$\sum\theta^{-1}\left[(s_g 1_G)(1_S g)\right]$$

$$=$$

$$\sum\theta^{-1}\,(s_g 1_G)\theta^{-1}(1_S g)$$

$$\because s_g\in\phi(I), \therefore (s_g 1_G)\in\theta(IG)$$

This implies that $\theta^{-1}(s_g 1_G)\in IG$

Also $\theta^{-1}(1_S g)\in RG$ and since IG is an ideal or RG

$$\therefore \theta^{-1}\,(s_g 1_G)\,\theta^{-1}(1_s g)\in IG$$

$$\Rightarrow \sum\theta^{-1}(s_g 1_G)\,\theta^{-1}(1_s g)\in IG$$

$$\Rightarrow\ \theta^{-1}\left(\sum s_g g\right)\in IG$$

Thus we see that $\theta^{-1}(\phi(I)G)\underline{C}\, IG$

Consider

$\psi: L_I(S)\rightarrow L_I(R)$,Then

$\psi(J)=\{r\in R \mid r1_G\ \in\theta^{-1}(JG)\}$ for every Ideal J of S

Claim : $\phi\ \psi$ =1, $\psi\ \phi$ =1

For any ideal I of R, consider $\psi\ \phi$(I). Take any r$\in\psi\ \phi$(I) Then r$\in$R and $r1_G\in\theta^{-1}(\phi(I)G)$ i.e. r$\in$ R and $r1_G\in IG$ and hence r $\in I$

$$\therefore \psi\phi(I)\underline{C}\, I$$

Conversely take any r$\in$I. Then r$\in\psi\phi(I)$ iff

r1$_G$$\in$ θ^{-1} $(\phi(I)G)$ i.e iff $\theta(r1_G)$ $\in\phi(I)G$

Now as θ induces an isomorphism between R and S, $\theta(r1_G) = s1_G$ for some $s \in S$ and so $s \in \phi(I) \quad (\because s1_G \in \theta(IG))$

$\therefore \theta(r1_G) \in \phi(I)G$

$\Rightarrow \quad r1_G \in \theta^{-1}(\phi(I)G)$

$\Rightarrow \quad r \in \psi\phi(I)$

$\therefore \quad I \subseteq \psi\phi(I)$

$\therefore \quad \psi\phi = 1$

Similarly $\phi\psi = 1$

Hence ϕ is a one-to-one correspondence provided that θ induces an isomorphism between R and S.

Remark 2.2 There are many instances when does induce an isomorphism between R and S. We mention a few :

(I) If R and S are P.I.D's which do not contain fields and <x> is an infinite cyclic group then R <x> ≅ S <x> implies that R ≅ S([2])

(ii) If R and S are integral domains and G a torsion abelian group then RG ≅ SG implies that R ≅ S ([3])

Example 2.3 Now we give an example to show that in case

θ: RG ≅ SG does not induce an isomorphism from R to S, the correspondence obtained by us need not be a one-to-one correspondence.

Let G_0, G_1, G_2, … be infinite cyclic groups and R=Z, S= ZG_0 and G=, $\pi_{i\geq 1} G_i$ the direct product of G_1, G_2,….. Then we have an isomorphism

θ: RG ≅ SG which does not induce an isomorphism between R and S.

Take J = $\Delta_Z(G_0)$ the augmentation ideal of ZG_0. J is an ideal of S, but there does not exist any ideal I of R such that $J = \phi(I)$.

Thus the correspondence θ is not a objective one.

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