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Parameter Estimation of Lomax-Poisson Distributionunder Type II Censored Samples

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ABSTRACT

In this paper, closed form expressions for the mode, mean deviations, and L-moments of Lomax-Poisson distribution are derived. Maximum likelihood estimators (MLE's) for the unknown parameters and the corresponding asymptotic variance covariance matrix of a Lomax-Poisson distributionare obtained under type II censored samples. Results obtained by Abd-Elfattah et al.(2013) in the complete case may be considered as a special case from present results. Based on Monte Carlo simulation study, comparisons are made through the different estimators by through their biases and mean squared error.

KEYWORDS

Lomax-Poisson distribution, maximum likelihood estimators, type II censoredsample, asymptotic variance covariance matrix.

1. Introduction

In analyzing lifetime data, new several probability distributions discussed by compounding well-known continuous and discrete distributions. The compounding approach gives new distributions that extend well-known families of distributions. The flexibility of such compound distributions come in terms of one or more failure rate which may be decreasing or increasing or bathtub shaped or unimodal shaped. The new three-parameter named as Lomax-Poissondistribution (abbreviated as LP), which is obtained by compounding the Lomax and zero truncated Poisson distributions. The compounding procedures follow the key idea proposed by Adamidis and Loukas (1998). The Lomax-Poisson distribution which may has decreasing failure rate hazard function, its application may cover a wide spectrum of areas ranging from biological, engineering sciences, economics,finance to medicine.

Let $W_1, W_2, ..., W_Z$ be independent identically Lomax random variables with the following probability density function

$$f(w;\theta,\gamma) = \frac{\theta}{\gamma}\left(1+\frac{w}{\gamma}\right)^{-(\theta+1)}, \quad w \succ 0, \theta, \gamma \succ 0$$

Where θ is the shape parameter, γ is the scale parameter and Z has a zero truncated Poisson distribution with parameter λ with the following probability mass function

$$P(Z=z;\lambda) = \frac{e^{-\lambda}\lambda^{z}}{\left(1-e^{-\lambda}\right)}\Gamma^{-1}(z+1), \quad \lambda \succ 0, z \in N = \{1,2,3,...\}$$

Where $\Gamma(.)$ is a gamma function. Assuming that the random variables Z and W are independent, define $X = \min(W_1, W_2, ..., W_Z)$, then the probability density function of Lomax-Poisson distribution take the form (Abd-Elfattah et al.(2013)):

$$f(x;\gamma,\theta,\lambda) = \frac{\theta\lambda}{\gamma\left(1-e^{-\lambda}\right)}\left(1+\frac{x}{\gamma}\right)^{-(\theta+1)} e^{-\lambda\left[1-\left(1+\frac{x}{\gamma}\right)^{-\theta}\right]}, \quad x \succ 0, \gamma, \theta, \lambda \succ 0 \tag{1}$$

With cumulative distribution function

$$F(x;\gamma,\theta,\lambda) = \frac{1}{\left(1-e^{-\lambda}\right)}\left[1-e^{-\lambda\left[1-\left(1+\frac{x}{\gamma}\right)^{-\theta}\right]}\right], \quad x \succ 0, \gamma, \theta, \lambda \succ 0 \tag{2}$$

The Lomax distribution is a limiting special case of Lomax-Poisson distribution. Abd-Elfattah et al. (2013) estimated the unknownparameters of the Lomax-Poisson distribution using maximum likelihood, moments, least square and weighted least squares estimation methods, and make an intensivenumerical study for evaluating the performance of parameter estimation.In Section 2, the mode, mean deviations, and L-moments are derived. In Section 3, the maximum likelihood estimators for the unknown parameters and the correspondingasymptotic variance covariance matrix are obtained under type II censored sample. Using simulation technique, the biases and mean squared errors in the estimation method are analyzed in Section 4.

2. Distributional Properties

In this section some properties of Lomax-Poisson distribution will be obtained such as mode, mean and median deviations, and L-moments.

(2.1) Mode

The mode of the Lomax-Poisson distribution is obtained by finding the first derivative of $\ln(f(x))$ with respect to x and equating it to zero. That is $\frac{d}{dx}\ln(f(x))=0$.

Taking logarithm for (1),

$$\ln f(x)=\ln\left(\frac{\lambda\theta}{\gamma(1-e^{-\lambda})}\right)-(\theta+1)\ln\left(1+\frac{x}{\gamma}\right)-\lambda\left[1-\left(1+\frac{x}{\gamma}\right)^{-\theta}\right].$$

Taking derivative with respect to x,

$$\frac{d\ln f(x)}{dx}=-\left(\frac{\theta+1}{\gamma}\right)\left(1+\frac{x}{\gamma}\right)^{-1}-\frac{\lambda\theta}{\gamma}\left(1+\frac{x}{\gamma}\right)^{-(\theta+1)}$$
$$=0,$$

The mode at $x=x_0$ is given by

$$x_0=\gamma\left[\left(\frac{-\lambda\theta}{\theta+1}\right)^{\frac{1}{\theta}}-1\right].$$

(2.2) Mean deviations

The mean deviation about the mean and the median are useful measures of variation for population. Let μ, M are the mean and median respectively of the Lomax-Poisson distribution.The mean deviation about the mean and the mean deviation about the median are, respectively, defined by

$$\delta_1(\mu) = 2\mu F(\mu) - 2\int_0^{\mu} x\, f(x)\, dx \tag{3}$$

$$\delta_2(M) = \mu - 2\int_0^{M} x\, f(x)\, dx \tag{4}$$

Theorem

Let X be a random variable distributed according to the Lomax-Poisson distribution. Then the mean deviation about the mean, δ_1 , and the mean deviation about the median, δ_2, are given as follows:

$$\delta_1(\mu) = \frac{2}{\left(1-e^{-\lambda}\right)}\left\{\begin{array}{l} \mu\left(1-e^{-\lambda\left[1-\left(1+\frac{\mu}{\gamma}\right)^{-\theta}\right]}\right) - \left(\gamma\,\theta\, e^{-\lambda}\right)\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{n!} \\ \times\left[\frac{\left[\left(1+\frac{\mu}{\gamma}\right)^{1-\theta(n+1)}-1\right]}{\left(1-\theta(n+1)\right)} + \frac{\left[\left(1+\frac{\mu}{\gamma}\right)^{-\theta(n+1)}-1\right]}{\left(\theta(n+1)\right)}\right] \end{array}\right\},$$

and

$$\delta_2(M) = \mu - \frac{\left(2\gamma\,\theta\, e^{-\lambda}\right)}{\left(1-e^{-\lambda}\right)}\left\{\sum_{n=0}^{\infty}\frac{\lambda^{n+1}}{n!}\left[\frac{\left[\left(1+\frac{M}{\gamma}\right)^{1-\theta(n+1)}-1\right]}{\left(1-\theta(n+1)\right)} + \frac{\left[\left(1+\frac{M}{\gamma}\right)^{-\theta(n+1)}-1\right]}{\left(\theta(n+1)\right)}\right]\right\}.$$

Proof

The proof follows by plugging the density function of the Lomax-Poisson distribution (1) into (3) and working out the integration I, where

$$I = \int_0^{\mu} x\, f(x)\, dx = \frac{\theta\lambda}{\gamma\left(1-e^{-\lambda}\right)}\int_0^{\mu} x\left(1+\frac{x}{\gamma}\right)^{-(\theta+1)} e^{-\lambda\left[1-\left(1+\frac{x}{\gamma}\right)^{-\theta}\right]}\, dx$$

Setting $y=\left(1+\frac{x}{\gamma}\right)$, so $dy=\left(\frac{1}{\gamma}\right)dx$ and using the expansion $e^{x}=\sum_{n=0}^{\infty}x^{n}/n!$, yields

$$I=\mu-\frac{\left(\lambda\gamma\theta e^{-\lambda}\right)}{\left(1-e^{-\lambda}\right)}\left\{\sum_{n=0}^{\infty}\frac{\lambda^{n}}{n!}\left[\frac{\left[\left(1+\frac{\mu}{\gamma}\right)^{1-\theta(n+1)}-1\right]}{\left(1-\theta(n+1)\right)}+\frac{\left[\left(1+\frac{\mu}{\gamma}\right)^{-\theta(n+1)}-1\right]}{\left(\theta(n+1)\right)}\right]\right\},$$

Substituting into relation (3) and manipulating the other terms gives directly the desired result. Similarly, the measure of mean deviation about median can be obtained.

(2.3) L-moments

Suppose $X_1, X_2, ..., X_n$ be a random sampleof size n fromthe Lomax-Poisson distribution in (1) and let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ denote the corresponding order statistics.Mokhtar (2015) introduced the probability density function of $X_{i:n}, 1\le i\le n$, as follows

$$f_{i:n}(x)=\frac{\theta\lambda}{\gamma B(i,n-i+1)}\left(1+\frac{x}{\gamma}\right)^{-(\theta+1)}e^{-\lambda\left[1-\left(1+\frac{x}{\gamma}\right)^{-\theta}\right]}$$
$$\times\sum_{j=0}^{n-i}\sum_{k=0}^{i+j-1}\binom{n-i}{j}\binom{i+j-1}{k}(-1)^{k+j}\left[\frac{1}{\left(1-e^{-\lambda}\right)^{i+j}}\left\{e^{-\lambda(k+1)\left[1-\left(1+\frac{x}{\gamma}\right)^{-\theta}\right]}\right\}\right],\quad 0\prec x\prec\infty$$

In what follows, we derive a general representation for the L-moments of X. The rth population L-moments is given by

$$E\left[X_{i:n}\right]=\int_{0}^{\infty}x\,f_{i:n}(x)\,dx$$

$$=\int_{0}^{\infty}x\left(\begin{array}{l}\frac{\theta\lambda}{\gamma B(i,n-i+1)}\left(1+\frac{x}{\gamma}\right)^{-(\theta+1)}e^{-\lambda\left[1-\left(1+\frac{x}{\gamma}\right)^{-\theta}\right]}\\ \times\sum_{j=0}^{n-i}\sum_{k=0}^{i+j-1}\binom{n-i}{j}\binom{i+j-1}{k}(-1)^{k+j}\left[\frac{1}{\left(1-e^{-\lambda}\right)^{i+j}}\left\{e^{-\lambda(k+1)\left[1-\left(1+\frac{x}{\gamma}\right)^{-\theta}\right]}\right\}\right]\end{array}\right)dx,$$

Let $y=\left(1+\frac{x}{\gamma}\right)$, so $x=\gamma(y-1)$ and $dy=\left(\frac{1}{\gamma}\right)dx$. After some transformation, we arrive to the formula:

$$E[X_{i:n}] = \gamma \sum_{l=0}^{\infty}\sum_{j=0}^{n-i}\sum_{k=0}^{i+j-1} \frac{(k+1)^l \lambda^{l+1} \zeta_{i,j,k,l}}{(l+1)![\theta(l+1)-1]}, \tag{5}$$

where

$$\zeta_{i,j,k,l} = \frac{n!(-1)^{k+j} e^{-\lambda-\lambda k}}{(i-1)!(n-i)!\left(1-e^{-\lambda}\right)^{i+j}} \binom{n-i}{j}\binom{i+j-1}{k}.$$

One readily can use the relation (5) to obtain the first L-moments of $X_{i:n}$. For example, we take $i = n = 1$ to obtain $L_1 = E[X_{1:1}]$which is the mean of the random variable X. as follows:

$$L_1 = E[X_{1:1}] = \frac{\gamma e^{-\lambda}}{\left(1-e^{-\lambda}\right)} \sum_{l=0}^{\infty} \frac{\lambda^{l+1}}{(l+1)![\theta(l+1)-1]},$$

This result is consistent with that obtained in Mokhtar (2015). The other two L-moments, L_2 and L_3 are respectively given by

$$L_2 = \frac{1}{2}\left[E[X_{2:2}] - E[X_{1:2}]\right]$$

$$= \gamma \left[\sum_{l=0}^{\infty}\sum_{k=0}^{1}\binom{1}{k} \frac{(-1)^k \lambda^{l+1}(k+1)^l e^{-\lambda-\lambda k}}{(l+1)!\left(1-e^{-\lambda}\right)^2[\theta(l+1)-1]} - \sum_{l=0}^{\infty}\sum_{j=0}^{1}\sum_{k=0}^{j}\binom{1}{j}\binom{j}{k} \frac{(-1)^{k+j}\lambda^{l+1}(k+1)^l e^{-\lambda-\lambda k}}{(l+1)!\left(1-e^{-\lambda}\right)^{1+j}[\theta(l+1)-1]} \right],$$

and

$$L_3 = \frac{1}{3}\left[E[X_{3:3}] - 2E[X_{2:3}] + E[X_{1:3}]\right]$$

$$= \gamma \left[\begin{array}{l} \displaystyle\sum_{l=0}^{\infty}\sum_{k=0}^{2}\binom{2}{k} \frac{(-1)^k \lambda^{l+1}(k+1)^l e^{-\lambda-\lambda k}}{(l+1)!\left(1-e^{-\lambda}\right)^3[\theta(l+1)-1]} - 2\sum_{l=0}^{\infty}\sum_{j=0}^{1}\sum_{k=0}^{j+1}\binom{1}{j}\binom{j+1}{k} \frac{2(-1)^{k+j}\lambda^{l+1}(k+1)^l e^{-\lambda-\lambda k}}{(l+1)!\left(1-e^{-\lambda}\right)^{2+j}[\theta(l+1)-1]} \\ \displaystyle + \sum_{l=0}^{\infty}\sum_{j=0}^{2}\sum_{k=0}^{j}\binom{2}{j}\binom{j}{k} \frac{(-1)^{k+j}\lambda^{l+1}(k+1)^l e^{-\lambda-\lambda k}}{(l+1)!\left(1-e^{-\lambda}\right)^{1+j}[\theta(l+1)-1]} \end{array} \right],$$

the method of L-moments consists of equating the first three L-moments of a population, $L_1, L_2,$ and L_3, to the corresponding L-moments of a sample, $l_1, l_2,$ and l_3, thus getting a number of equations that are needed to be solved, numerically, in terms of the unknown parameters.

3. Maximum likelihood estimators for Type II Censored Sample In a typical life test, *n* specimens are placed under observation and as each failure occurs the time isnoted. When a predetermined total number of failures r is reached, the test is terminated. In this case the data collected consist of observations $x_{(1)} < x_{(2)} < ... < x_{(r)}$ plus the information that (n-r) items survived beyond the time of termination $x_{(r)}$, this censoring scheme is known as type II censoring and also the collected data is saidto be censored type II data. Cohen (1965) gave the likelihood function for type II censoring:

$$L = C\prod_{i=1}^{r} f(x_{(i)};\theta)\,[1-F(x_{(r)};\theta)]^{n-r}$$

where C is a constant, r is the number of uncensored sample, $x_{(r)}$ is the observed value of the *ith* order statistic, $f(x_{(i)};\theta)$ and $F(x_{(r)};\theta)$ are the density function and the cumulative function of the underlyingdistribution, respectively.

For the Lomax-Poisson distribution (1), the likelihood function will be

$$L(x;\theta,\gamma,\lambda) = C\left(\frac{\theta\lambda}{\gamma\left(1-e^{-\lambda}\right)}\right)^{r}\prod_{i=1}^{r}\left[\left(1+\frac{x_{(i)}}{\gamma}\right)^{-(\theta+1)} e^{-\lambda\left[1-\left(1+\frac{x_{(i)}}{\gamma}\right)^{-\theta}\right]}\right].\left[1-\frac{1}{\left(1-e^{-\lambda}\right)}\left\{1-e^{-\lambda\left[1-\left(1+\frac{x_{(i)}}{\gamma}\right)^{-\theta}\right]}\right\}\right]^{n-r} \tag{6}$$

Taking the logarithm, (6) becomes

$$\ln L \propto r\ln\theta + r\ln\lambda - r\ln\gamma - r\ln\left(1-e^{-\lambda}\right) - (\theta+1)\sum_{i=1}^{r}\ln\left(1+\frac{x_{(i)}}{\gamma}\right) - \lambda\sum_{i=1}^{r}\left[1-\left(1+\frac{x_{(i)}}{\gamma}\right)^{-\theta}\right]$$
$$-(n-r)\ln\left\{1-e^{-\lambda\left[1-\left(1+\frac{x_{(r)}}{\gamma}\right)^{-\theta}\right]}\right\} + (n-r)\ln\left(1-e^{-\lambda}\right),$$

$$\ln L \propto r\ln\theta + r\ln\lambda - r\ln\gamma - r\ln\left(1-e^{-\lambda}\right) - (\theta+1)\sum_{i=1}^{r}\ln b_i - \lambda\sum_{i=1}^{r}\left[1-(b_i)^{-\theta}\right] \tag{7}$$
$$-(n-r)\ln\{1-d_r\} + (n-r)\ln\left(1-e^{-\lambda}\right),$$

where $b_i = \left(1+\frac{x_{(i)}}{\gamma}\right)$ and $d_i = e^{-\lambda\left[1-(b_i)^{-\theta}\right]}$. Differentiate (7) with respect to the unknown parameters and equating to zero:

$$\frac{\partial \ln L}{\partial \gamma} = \frac{-r}{\gamma} + \left(\frac{\theta+1}{\gamma^2}\right)\sum_{i=1}^{r}\left(\frac{x_{(i)}}{b_i}\right) + \frac{\lambda\theta}{\gamma^2}\sum_{i=1}^{r}\left[\frac{x_{(i)}}{(b_i)^{\theta+1}}\right] - \left(\frac{(n-r)\lambda\theta}{\gamma^2}\right)\frac{d_r\,x_{(r)}}{(b_r)^{\theta+1}\{1-d_r\}} = 0,$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{r}{\theta} - \sum_{i=1}^{r} \ln b_i - \lambda \sum_{i=1}^{r} \left[\frac{\ln b_i}{(b_i)^{\theta}} \right] - (n-r) \frac{\lambda d_r \ln(b_r)}{b_r^{\theta} \{1-d_r\}} = 0 ,$$

and

$$\frac{\partial \ln L}{\partial \lambda} = \frac{r}{\lambda} - \frac{(r e^{-\lambda})}{(1-e^{-\lambda})} - \sum_{i=1}^{r} \left[1-(b_i)^{-\theta}\right] - (n-r) \frac{(1-(b_r)^{-\theta}) d_r}{\{1-d_r\}} + (n-r) \left(\frac{e^{-\lambda}}{1-e^{-\lambda}} \right) = 0.$$

Then, the maximum likelihood estimates of the parameters γ, θ and λ can be obtained by solving the above system of equations. No explicit form for these parameters, we use anumerical technique using Mathcad2001 Package to obtain $\hat{\gamma}, \hat{\theta}$ and $\hat{\lambda}$.

The asymptotic variance-covariance matrix for the estimators $\hat{\gamma}, \hat{\theta}$ and $\hat{\lambda}$ can be obtained byinverting the information matrix with the elements that are negative of the expected values of the second orderderivatives of logarithm of the likelihood function. The elements of the asymptotic variance covariance matrixcan be approximated by Cohen's (1965) result:

$$-E\left(\frac{\partial^2 \ln L(\underline{\omega})}{\partial \omega_i \partial \omega_j} \right) \approx - \left. \frac{\partial^2 \ln L(\underline{\omega})}{\partial \omega_i \partial \omega_j} \right|_{\underline{\omega} = \hat{\underline{\omega}}}$$

As follows

$$\left. \frac{-\partial^2 \ln L}{\partial \gamma^2} \right|_{\substack{\gamma=\hat{\gamma} \\ \theta=\hat{\theta} \\ \lambda=\hat{\lambda}}} = \frac{-r}{\hat{\gamma}^2} + \left(\frac{\hat{\theta}+1}{\hat{\gamma}^3} \right) \sum_{i=1}^{r} \left(\frac{x_{(i)}}{\hat{b}_i} \left[2 - \frac{x_{(i)}}{\hat{\gamma} \hat{b}_i} \right] \right) + \frac{\hat{\lambda} \hat{\theta}}{\hat{\gamma}^3} \sum_{i=1}^{r} \left[\frac{x_{(i)}}{\left(\hat{b}_i \right)^{\hat{\theta}+1}} \left(2 - \frac{\left(\hat{\theta}+1 \right) x_{(i)}}{\hat{\gamma} \hat{b}_i} \right) \right]$$

$$+ \left(\frac{(n-r) \hat{\lambda} \hat{\theta} \hat{d}_r x_{(r)}}{\hat{\gamma}^3 \left(\hat{b}_r \right)^{\hat{\theta}+1} \left(1 - \hat{d}_r \right)} \right) \left[2 - \frac{\hat{\theta} \hat{\lambda} x_{(r)}}{\hat{\gamma} \left(\hat{b}_r \right)^{\hat{\theta}+1}} - \frac{\left(\hat{\theta}+1 \right) x_{(r)}}{\hat{\gamma} \hat{b}_r} - \frac{\hat{\lambda} \hat{\theta} \hat{d}_r x_{(r)}}{\hat{\gamma} \left(\hat{b}_r \right)^{\hat{\theta}+1} \left(1 - \hat{d}_r \right)} \right],$$

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$$\left.\frac{-\partial^2 \ln L}{\partial\gamma\,\partial\lambda}\right|_{\substack{\gamma=\hat{\gamma}\\ \theta=\hat{\theta}\\ \lambda=\hat{\lambda}}} = \frac{-\hat{\theta}}{\hat{\gamma}^2}\sum_{i=1}^{r}\left[x_{(i)}\,\hat{b}_i^{\,-\hat{\theta}-1}\right]-\left(\frac{(n-r)\hat{\theta}\hat{d}_r\,x_{(r)}}{\hat{\gamma}^2\left(\hat{b}_r\right)^{\hat{\theta}+1}\left(1-\hat{d}_r\right)}\right)\left[1-\hat{\lambda}\left(1-\hat{b}_r^{\,-\hat{\theta}}\right)-\frac{\hat{\lambda}\hat{d}_r\left(1-\hat{b}_r^{\,-\hat{\theta}}\right)}{\left(1-\hat{d}_r\right)}\right],$$

$$\left.\frac{-\partial^2 \ln L}{\partial\gamma\,\partial\theta}\right|_{\substack{\gamma=\hat{\gamma}\\ \theta=\hat{\theta}\\ \lambda=\hat{\lambda}}} = \frac{-1}{\hat{\gamma}^2}\sum_{i=1}^{r}\left(\frac{x_{(i)}}{\hat{b}_i}\right)+\frac{\hat{\lambda}}{\hat{\gamma}^2}\sum_{i=1}^{r}\left[\frac{x_{(i)}}{\left(\hat{b}_i\right)^{\hat{\theta}+1}}\left(\hat{\theta}\ln\left(\hat{b}_i\right)-1\right)\right]$$

$$-\left(\frac{(n-r)\hat{\lambda}\hat{d}_r\,x_{(r)}}{\hat{\gamma}^2\left(\hat{b}_r\right)^{\hat{\theta}+1}\left(1-\hat{d}_r\right)}\right)\left[1-\hat{\theta}\ln\left(\hat{b}_r\right)-\frac{\hat{\lambda}\hat{\theta}\ln\left(\hat{b}_r\right)}{\hat{b}_r^{\,\hat{\theta}}}-\frac{\hat{\lambda}\hat{\theta}\hat{d}_r\ln\left(\hat{b}_r\right)}{\left(\hat{b}_r\right)^{\hat{\theta}}\left(1-\hat{d}_r\right)}\right],$$

$$\left.\frac{-\partial^2 \ln L}{\partial\theta^2}\right|_{\substack{\gamma=\hat{\gamma}\\ \theta=\hat{\theta}\\ \lambda=\hat{\lambda}}} = \frac{r}{\hat{\theta}^2}-\hat{\lambda}\sum_{i=1}^{r}\left[\left(\hat{b}_i\right)^{-\hat{\theta}}.\left(\ln\hat{b}_i\right)^2\right]-(n-r)\frac{\hat{\lambda}\left(\ln\hat{b}_r\right)^2\hat{d}_r}{\left(\hat{b}_r\right)^{\hat{\theta}}\left\{1-\hat{d}_r\right\}}\left[1+\hat{\lambda}\left(\hat{b}_r\right)^{-\hat{\theta}}+\frac{\hat{\lambda}\hat{d}_r}{\left(\hat{b}_r\right)^{\hat{\theta}}\left(1-\hat{d}_r\right)}\right],$$

$$\left.\frac{-\partial^2 \ln L}{\partial\theta\,\partial\lambda}\right|_{\substack{\gamma=\hat{\gamma}\\ \theta=\hat{\theta}\\ \lambda=\hat{\lambda}}} = \sum_{i=1}^{r}\left[\ln\left(\hat{b}_i\right)\hat{b}_i^{\,-\hat{\theta}}\right]+\left(\frac{(n-r)\hat{d}_r\ln\left(\hat{b}_r\right)}{\left(\hat{b}_r\right)^{\hat{\theta}}\left(1-\hat{d}_r\right)}\right)\left[1-\hat{\lambda}\left(1-\hat{b}_r^{\,-\hat{\theta}}\right)-\frac{\hat{\lambda}\hat{d}_r\left(1-\hat{b}_r^{\,-\hat{\theta}}\right)}{\left(1-\hat{d}_r\right)}\right],$$

and

$$\left.\frac{-\partial^2 \ln L}{\partial\lambda^2}\right|_{\substack{\gamma=\hat{\gamma}\\ \theta=\hat{\theta}\\ \lambda=\hat{\lambda}}} = \frac{r}{\hat{\lambda}^2}-\frac{\left(r\,e^{-\hat{\lambda}}\right)}{\left(1-e^{-\hat{\lambda}}\right)^2}-(n-r)\frac{\left(1-\left(\hat{b}_r\right)^{-\hat{\theta}}\right)^2\hat{d}_r}{\left\{1-\hat{d}_r\right\}}\left[1+\frac{\hat{d}_r}{\left(1-\hat{d}_r\right)}\right]+(n-r)\frac{e^{-\hat{\lambda}}}{\left(1-e^{-\hat{\lambda}}\right)^2}.$$

Again, a numerical technique using Mathcad2001 Package and computer facilities are used to obtain thevariance-covariance matrix.

4. Simulation study:

It is very difficult to compare the theoretical performance of the different estimators.Therefore, simulation is needed to compare the performances of the different estimators, mainly, with respect to their biases and mean square errors (MSEs) for different sample sizes. A numericalstudy is performed using Mathcad (2001) software. Different sample sizes are considered through the experiment $n = 40(10)100$. In addition, the values of the parameters are considered, $\gamma = 0.8, \theta = 1.2$ and $\lambda = 0.6$. The experiment will be repeated 1000 times. In each experiment, the biases and MSEs for different maximum likelihood estimators under type II censored samples will be reported in Table 1. An algorithm for obtaining estimates can be described in the following steps:

Step (1): Generate a random sample of size n, $x_1, x_2, ..., x_n$ from Lomax-Poisson distribution (1) by using the following transformation $x_{i,j} = \gamma \left\{ \left[\frac{1}{\lambda} \ln\left[1 - u_{i,j}\left(1 - e^{-\lambda}\right)\right] + 1 \right]^{\frac{-1}{\theta}} - 1 \right\}$, $i, j = 1,2,...,n$ and $u_{i,j}$ are random sample from uniform (0,1).

Step (2): Sort the random sample $x_1, x_2, ..., x_n$ to obtain the ordered sample $x_{(1)}, x_{(2)}, ..., x_{(n)}$. Eliminate the elements from the sample that are outside the censoring level r to obtain a censored sample.

Step (3): Use the censored sample obtained in step (2), to obtain the maximum likelihood estimators $\hat{\gamma}, \hat{\theta}$ and $\hat{\lambda}$ in case of type II censored, clearly complete samples may be considered as a special case of censored samples.

Step (4): Steps from 1to 3 will be repeated 1000 times for each sample size. Then, the biases and MSEs of different estimators of unknown parameters are computed.

From Table, the following conclusions can be observed on the properties of estimated parameters from Lomax-Poisson distribution

1- In complete case, all cases considered the biases and MSEs of the different estimators of γ, θ and λ decreases as sample size increases.
2- For type II censored, the biases and MSEs of the estimators of γ and θ decrease for all sample sizes and the estimator of λ for sample size 100.When *r=n* the results reduce to complete sample case [result of Abd-Elfattah (2013) as a special case].

In general, it is observed thatthe maximum likelihood estimates of the parametersunder type II censored is the best in almost cases considered for large sample (n=100) with respect to bias and MSE for estimating γ, θ and λ of the Lomax-Poisson distribution.

Table: Results of simulation study of Biases and MSEs of the maximum likelihood estimates of the parameters $(\gamma, \theta, \lambda)$ based on type II censored samples for the Lomax-Poisson distribution.

Sample size 'n'	Scheme	r	$\hat{\gamma}$		$\hat{\theta}$		$\hat{\lambda}$	
			Bias	MSE	Bias	MSE	Bias	MSE
40	Type II	30	0.036	0.003	-0.364	0.323	0.67	1.319
		35	0.007	0.001	-0.392	0.254	0.577	1.675
	Complete	40	-0.008	0.0009	-0.408	0.239	0.571	1.541
50	Type II	30	0.085	0.015	-0.315	0.508	0.535	0.664
		40	0.022	0.002	-0.378	0.289	0.628	1.546
	Complete	50	-0.009	0.0007	-0.409	0.234	0.505	1.479
60	Type II	40	0.057	0.007	-0.343	0.398	0.62	0.916
		50	0.013	0.001	-0.387	0.267	0.552	1.598
	Complete	60	-0.01	0.0006	-0.41	0.233	0.481	1.418
70	Type II	50	0.045	0.005	-0.355	0.352	0.643	1.108
		60	0.009	0.001	-0.391	0.251	0.472	1.534
	Complete	70	-0.009	0.0005	-0.41	0.228	0.475	1.328
100	Type II	80	0.019	0.001	-0.381	0.272	0.498	1.364
		90	0.0002	0.0004	-0.4	0.226	0.295	1.353
	Complete	100	-0.01	0.0003	-0.41	0.219	0.461	1.221

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