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Analysis of Some Results on Fuzzy Continuous Mapping

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ABSTRACT

In this paper we analyze some properties of fuzzy continuous mapping , fuzzy semi continuous mapping and fuzzy almost continuous after giving the fundamental definitions

KEYWORDS Topology, fuzzy open set, fuzzy closed set, fuzzy continuity, fuzzy semi continuity, fuzzy almost continuous.

Introduction: The usual notion of a set was generalized with the introduction of fuzzy sets by Zadeh in his classical paper [1] of 1965.This was further generalized by Goguen in [2] where he introduced L- fuzzy sets. Algebraic properties of these sets were studied by them and de Luca and Termini[3].Because the concept of fuzzy sets corresponds to the physical situation in which there is no precisely defined criterion for membership, fuzzy sets are having useful and increasing application in various fields including probabilities theory, information theory and pattern recognition. In accordance with this, fuzzy topological vector spaces were introduced by Chang[4], Lowen[5], Katsaras[6], Rosenfeld[7], Kastaras and Liu[8], Foster[9] respectively.

As the study of general topology can be regarded as a special case of fuzzy topology where all fuzzy sets in question take values 0 and 1 only several workers continued investigations in fuzzy topological spaces. This paper is also devoted to the study of fuzzy topological spaces with specific attention to the weaker forms of fuzzy continuity. Here we generalizations of semi continuous mapping and almost continuous mapping and weakly continuous mapping in fuzzy setting.

Fundamental properties of Fuzzy continuous mapping

Definition 1.1. Given fuzzy topological space $(X,\tau) and\ (Y,\gamma)$ a function $f: X \to Y$ is fuzzy continuous if the inverse image under f of any open fuzzy set in Y is an open fuzzy set in X; that is $f^{-1}(\vartheta) \epsilon\ \tau$ whenever $\vartheta \in \gamma$.

Proposition 1.2. (a) The identity $id_X : (X,\tau) \to (X,\tau)$ on a fuzzy topological space (X,τ) is fuzzy continuous.

(b) A composition of fuzzy continuous functions is fuzzy continuous.

Proof. (a) For $\vartheta \in \tau$; $id_X^{-1}(\vartheta) = \vartheta \text{ o } id_X = \vartheta$

(b) Let $f:(X,\tau) \to (Y,\gamma) and\ g:(X,\gamma) \to (Y,\beta)$ be fuzzy continuous.

For $\mu \epsilon \beta$, $(gof)^{-1}(\mu) = \mu o(gof) = (\mu og)of = f^{-1}(\mu og) = f^{-1}(g^{-1}(\mu))$. $g^{-1}(\mu) \in \gamma$. Since g is fuzzy continuous, and so $(gof)^{-1}(\mu) = f^{-1}(g^{-1}(\mu)) \in \tau$ since f is fuzzy continuous.

Proposition1.3: Let (X,τ) be fuzzy topological space. Then every constant function from (X,τ) into another fuzzy topological space is fuzzy continuous iff τ contains all constant fuzzy sets in X.

Proof: Suppose that every constant function from (X,τ) into any fuzzy topological space is fuzzy continuous and consider the fuzzy topology γ on $[0,1]$ defined by $\gamma = \{\overline{0}, \overline{1}, id_{[0,1]}\}$. Let k be a real number, $0 \le k \le 1$. The constant function $f: X \to [0,1]$ defined by $f(x) = k\ for\ every\ x \in X$, is fuzzy continuous and so $f^{-1}(id_{[0,1]}) \in \tau$. But for $x \in X$, $f^{-1}(id_{[0,1]})(x) = id_{[0,1]}(f(x)) = id_{[0,1]}(k) = k$, when the constant fuzzy set k in X $\in \tau$. Conversely, Suppose t τ hat τ contains all constant fuzzy sets in X and consider a constant function $f:(X,\tau) \to (Y,\gamma)$ defined by $f(x) = y_0$. If $\vartheta \in \gamma$, then for any $x \in X$.

we have $f^{-1}(\vartheta)(x) = \vartheta(f(x)) = \vartheta(y_0)$, so that $f^{-1}(\vartheta)$ is a constant fuzzy set in X and hence, a member of τ. Thus f is fuzzy continuous.

Fuzzy Semicontinuity:

Let φ be a fuzzy set of a fuzzy space (X,τ)

Definition 2.1. φ is called (i) a fuzzy semiopen set X of if there exists a $\vartheta \in \tau$ such that $\vartheta \le \varphi \le Cl\ \vartheta$, and (ii) a fuzzy semiclosed set of X if there exists a $\vartheta^{/} \in \tau$such that $int\ \vartheta \le \varphi \le \vartheta$.

Definition 2.2 .Let $f:(X,\tau) \to (Y,\gamma)$ be a mapping from a fuzzy space X to another fuzzy space Y. f is called, (i) a fuzzy semicontinuous mapping, if $f^{-1}(\varphi)$ is a fuzzy semiopen set of X, for each $\varphi \in \gamma$.

Example 2.3. (a) A fuzzy semicontinuous mapping need not be a fuzzy continuous mapping. Let (I,τ) be a fuzzy space and let $f: I \to I$ be defined by $f(x) = x/2$. Simple computation give $f^{-1}(0) = 0$, $f^{-1}(1) = 1$, $f^{-1}(\mu_1) = 0$ and $f^{-1}(\mu_2) = \mu_1^{/} = f^{-1}(\mu_1 \cup \mu_2)$. Because Cl $\mu_2 = \mu_1^{/}$, $\mu_1^{/}$ is a fuzzy semiopen set; hence f is a fuzzy semicontinuous mapping. As $\mu_1^{/} \notin \tau$, f is not a fuzzy continuous mapping.

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Lemma 2.4. Let $f: X \to Y$ be a mapping and $\{\lambda_\alpha\}$ be a family of fuzzy sets of Y, then (a) $f^{-1}(\cup \lambda_\alpha) = \cup f^{-1}(\lambda_\alpha)$ and (b) $f^{-1}(\cap \lambda_\alpha) = \cap f^{-1}(\lambda_\alpha)$.

Lemma 2.5. For mappings $f_i: X_i \to Y_i$ and fuzzy sets λ_i of $Y_{i,}$ $i = 1,2$;
we have $(f_1 \times f_2)^{-1}(\lambda_1 \times \lambda_2) = f_1^{-1}(\lambda_1) \times f_2^{-1}(\lambda_2)$.

Theorem 2.6. Let$X_1, X_2, Y_1, and\ Y_2$ be fuzzy spaces such that X_1 is product related to X_2. Then, the product $f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ of fuzzy semicontinuous mapping $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ is fuzzy semicontinuous.

Proof: Let $\lambda \equiv \cup (\lambda_\alpha \times \mu_\beta)$, where λ_α's and μ_β 's are fuzzy open set of Y_1 and Y_2
Respectively, be a fuzzy open set of $Y_1 \times Y_2$. Using lemma 2.4(a) and 2.5, we have
$(f_1 \times f_2)^{-1}(\lambda) = \cup [f_1^{-1}(\lambda_\alpha) \times f_2^{-1}(\mu_\beta)]$.
That $(f_1 \times f_2)^{-1}(\lambda)$ is a fuzzy semiopen set since any union of fuzzy semiopen sets is a fuzzy semiopen set.

Theorem 2.7. Let $X, X_1,$ and X_2 be fuzzy spaces and $p_i: X_1 \times X_2 \to X_i$ $(i = 1,2)$ be the projection of $X_1 \times X_2$ onto X_i. Then , if $f: X \to X_1 \times X_2$ is a fuzzy semi continuous mapping, $p_i\, f$ is also fuzzy semicontinuous.

Proof: For a fuzzy open set μ of X_i, we have $(p_i f)^{-1}(\mu) = f^{-1}(p_i{}^{-1}(\mu))$. That p_i is a fuzzy continuous mapping [11] and f is a fuzzy semicontinuous mapping imply that $(p_i f)^{-1}(\mu)$ is a fuzzy semiopen set of X.

Lemma 2.8. Let $g: X \to X \times Y$ be the graph of a mapping $f: X \to Y$. Then , if λ is a fuzzy set of X and μ is a fuzzy set of Y, $g^{-1}(\lambda \times \mu) = \lambda \cap f^{-1}(\mu)$.

Theorem: 2.9. Let $f: X \to Y$ be a mapping from a fuzzy space X to another fuzzy space Y. Then if the graph $g: X \to X \times Y$ of f is fuzzy semicontinuous, f is also fuzzy semicontinuous.

Proof: Using lemma 2.8 , $f^{-1}(\mu) = 1 \cap f^{-1}(\mu) = g^{-1}(1 \times \mu)$, for each fuzzy open set μ of Y . Since g is a fuzzy semicontinuous mapping and $1 \times \mu$ is a fuzzy open set $X \times Y$, $f^{-1}(\mu)$ is a fuzzy semiopen set X of and hence f is a fuzzy semi continuous mapping.

Fuzzy Almost Continuous Mapping:

Definition3.1. A fuzzy set λ of a fuzzy space X is called (i) a fuzzy regular open set of X if $Int\ Cl\ \lambda = \lambda$, and (ii) a Fuzzy regular Closed set of X if $Cl\ Int\ \lambda = \lambda$.

Definition 3.2. Let $f: (X, \tau) \to (Y, \gamma)$ be a mapping from a fuzzy space X to another fuzzy space Y. f is called, (i) a fuzzy almost continuous mapping, if $f^{-1}(\lambda) \in \tau$ for each fuzzy regular open set λ of Y.

Lemma 3.3 (a) A fuzzy set $\lambda$of a fuzzy space X is fuzzy regular open iff $\lambda^{/}$ is fuzzy regular closed.

(b) The interior of a fuzzy closed set is a fuzzy regular open set.

Theorem 3.4. let $f:(X,\tau)\rightarrow(Y,\gamma)$ be amapping . Then the following are equivalent:

(i) f is a fuzzy almost continuous mapping.

(ii) $f^{-1}(\mu)$ is a fuzzy closed set, for each fuzzy regular closed set μ of Y,

(iii) $f^{-1}(\lambda)\leq Int\, f^{-1}(Int\, Cl\, \lambda)$, for each fuzzy open set λ of Y.

(iv) $Cl\, f^{-1}(Cl\, Int\, \mu)\leq f^{-1}(\mu)$, for each fuzzy closed set μ of Y,

Proof: $f^{-1}(\lambda^{/})$=$(f^{-1}(\lambda))^{/}$, for any fuzzy set λ of Y, (i) ⇔(ii) follows from lemma 3.3(a).

(i) ⇔(iii). Since λ is a fuzzy open set of Y , $\lambda\leq Int\, Cl\, \lambda$ and hence $f^{-1}(\lambda)\leq f^{-1}(Int\, Cl\, \lambda)$, By lemma 3.3 (b) , $Int\, Cl\, \lambda$ is a fuzzy regular open set of Y, hence $f^{-1}(Int\, Cl\, \lambda)$ is a fuzzy open set of X . Thus $f^{-1}(\lambda)\leq Int\, f^{-1}(Int\, Cl\, \lambda)$.

(iii) ⇔(i) Let λ be a fuzzy regular open set of Y, then we have $f^{-1}(\lambda)\leq Int\, f^{-1}(Int\, Cl\, \lambda)=Int\,(f^{-1}(\lambda)$. Thus $f^{-1}(\lambda)=Int\ f^{-1}(\lambda)$ shows that $f^{-1}(\lambda)$ is a fuzzy open set of X.

(ii) ⇔(iv) can similarly be proved.

Remark 3.5. Definition 3.2 is a generalization of almost continuous maping in the sense of Singal and Singal[10] in the fuzzy setting and theorem 3.4 generalizes (a), (b), (e) and (f) of theorem2.2 in [10].

Remark 3.6. Clearly a fuzzy continuous mapping is a fuzzy almost continuous mapping. That the converse need not be true is showen by following example.Also, the example shows that a fuzzy almost continuous mapping need not be a fuzzy semicontinuous mapping.

(a) Example:Let $\lambda,\mu\ and\ \vartheta$ be fuzzy sets of I defined as follws: for each $x\epsilon I$,

$$\lambda(x)=x,\qquad \mu(x)=1-x,\qquad \nu(x)=\begin{Bmatrix} x, & if\ 0\leq x\leq\frac{1}{2} \\ 0, & if\ \frac{1}{2}<x\leq 1 \end{Bmatrix}$$

Consider fuzzy topologies $\tau_1=\{0,\ \lambda,\ \mu,\ \lambda\cup\mu,\ \lambda\cap\mu,\ 1\}$ and $\tau_2=\{0,\ \lambda,\ \mu,\ \lambda\cup\mu,\ \lambda\cap\mu,$ $1\}$ on I and the mapping $i:(I,\tau_1)\rightarrow(I,\tau_2)$ defined by $i(x)=x$, for each $x\in I$. It is clear that $\lambda,\ \mu,\ \lambda\cup\mu\ and\ \lambda\cap\mu$ being both fuzzy open and fuzzy closed are fuzzy regular open sets of (I,τ_2), while ν is not. Nothing that $\nu\epsilon\,\tau_1$, it is obvious that i is a fuzzy almost continuous mapping which is not fuzzy continuous.

Also, because 0 is the only fuzzy open set contained in ν, $\nu=i^{-1}(\nu)$ is not a fuzzy semiopen set of (I,τ_1) and hence i is not a fuzzy semicontinuous mapping.

(b) Example: A fuzzy semicontinuous mapping need not be a fuzzy continuous mapping . Refer to example 2.3(a). The mapping f is fuzzy semicontinuous but is not fuzzy almost continuous for

$\mu_1^{/} = f^{-1}(\mu_2) \nleq \mu_2 = Int f^{-1}(Int\ Cl\ \mu_2)$.

Examples 3.6(a) and 3.6(b) establish the following results:

Theorem 3.7. Fuzzy semicontinuity and fuzzy almost continuity are independent notions.

Definition 3.8. A fuzzy space (X, τ) is called a fuzzy semiregular space iff the collections of all fuzzy regular open sets of X forms a base for fuzzy topology τ.

Lemma: 3.9. For a family $\{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X,$\cup\ Cl\lambda_\alpha \leq Cl\ \cup\ \lambda_\alpha$. In case A is a finite set, $\cup\ Cl\lambda_\alpha \leq Cl\ \cup\ \lambda_\alpha$. Also $\cup\ Int\lambda_\alpha \leq Int\ \cup\ \lambda_\alpha$.

Theorem 3.10. Let f: $(X, \tau) \rightarrow (Y, \gamma)$ be a mapping be a mapping from a fuzzy space $X$to a fuzzy semiregular space Y. Then f is fuzzy almost continuous iff f is fuzzy continuous.

Proof: Due to Remark 3.6, it suffices to show that if f is fuzzy almost continuous then it is fuzzy continuous. Let $\lambda \in \gamma$, $then\ \lambda = \cup\ \lambda_{\alpha,}$ where $\lambda_{\alpha,}$'s are fuzzy regular open sets of Y. Now , using lemma 2.4 (a), 3.9 and theorem 3.4 (iii), we get $f^{-1}(\lambda) \leq \cup\ f^{-1}(\lambda_\alpha) \leq \cup\ Int\ f^{-1}(Int\ Cl\lambda_\alpha) = \cup\ Int\ f^{-1}(\lambda_\alpha) \leq Int\ \cup f^{-1}(\lambda_\alpha) = Int\ f^{-1}(\lambda)$, which shows that $f^{-1}(\lambda) \in \tau$.

Conclusion: We have analysed fuzzy continuity, fuzzy semi continuity and fuzzy almost continuous mapping and have investigated its different properties. The results of this article can be applied for further investigations and applications in studing different properties weak continuity of functions.

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