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On Fixed Point Theorems in Fuzzy 2- Metric Space

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ABSTRACT

In this research article we inaugurate a fixed point theorem in fuzzy 2-metric space by using the conditions of R weakly commuting of type (Ag) and (E.A) property and further to discuss the existence of fixed point in the non-compatible maps.

KEYWORDS Fixed point, fuzzy 2- metric space , (E.A) property, R- weakly commuting.

1. Introduction:

The theory of fuzzy sets was presented by Zadeh[19] in 1965. To practice this theory in topology and exploration, several authors have widely developed the concept of fuzzy sets and then its applications. Fixed point propositions in fuzzy mathematics are developing with dynamic hope and vital confidence. Using the theory of fuzzy sets, the fuzzy metric space was presented by Kramosil and Michalek[10] in 1975. Grabiec[7] ascertained the reduction standard in fuzzy metric space in 1988. In addition, George and Veeramani[6] improved the concept of fuzzy metric space by the assistance of t-norms in 1994. The theory of 2-metric space was introduced by Gahler [5]. He deals the area of function in Euclidian space. Iseki and et.al[9] initiated to prove contraction type mapping in 2-metric space. Cho[4]. Kutukcu and et.al[12] proved a common fixed point theorem for three mappings in fuzzy 2-metric space. Sanjaykumar[11] discussed the concept of fuzzy 2-metric space akin to 2-metric space introduced by Gahler.

In this research paper present non-compatible point wise R-weakly commuting self maps of fuzzy 2-metric space and proofs were discussed.

2. Preliminaries

In this segment we recall some descriptions and acknowledged results in fuzzy 2-metric space.

Definition 2.1 A binary operation * : $[0,1] \times [0,1] \to [0,1]$ is called a *continuous t-norm* if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2 The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, 0) = 0,$

(ii) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$,

(iii) $M(x,y,t) = M(y,x,t)$,

(iv) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$,

(v) $M(x,y,\cdot):[0,\infty) \to [0,1]$ is left continuous for all $x,y,z \in X$ and $t,s > 0$.

Note that $M(x,y,t)$ can be thought of as the degree of nearness between x and y with respect to t.

Definition 2.3 The 3-tuple $(X,M,*)$ is called a fuzzy 2-metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^3 \times [0,\infty)$ satisfying the following conditions : for all $x,y,z,u \in X$ and $t_1,t_2,t_3 > 0$,

(i) $M(x,y,z,0) = 0$,

(ii) $M(x,y,z,t) = 1$ for all $t > 0$ if and only if atleast two of the three points are equal,

(iii) $M(x,y,z,t) = M(x,z,y,t) = M(y,z,x,t)$

for all $t > 0$, (Symmetry about first three variables)

(iv) $M(x,y,z,t_1+t_2+t_3) \geq M(x,y,u,t_1) * M(x,u,z,t_2) * M(u,y,z,t_3)$,

(This corresponds to tetrahedron inequality in fuzzy 2-metric space. The function value $M(x,y,z,t)$ may be interpreted as the probability that the area of triangle is less than t.)

(v) $M(x,y,z,.):[0,1) \to [0,1]$ is left continuous.

Definition 2.4 Self mappings A and B of a fuzzy 2-metric space $(X,M,*)$ is said to be compatible if $\lim_{n\to\infty} M(ABx_n,BAx_n,a,t) = 1$ for all $a \in X$ and $t > 0$, whenever $\{x_n\}$is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$$

for some $z \in X$.

From the above definition it is inferred that A and B are non-compatible maps from fuzzy 2-metric space $(X,M,*)$ into itself if $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$, but either $\lim_{n\to\infty} M(ABx_n,BAx_n,a,t) \neq 1$ or the limit does not exist.

Definition 2.5 A pair of self-mappings (A.S) of a fuzzy 2-metric space $(X,M,*)$ is said to be

(i) Weakly commuting if $M(ASx, SAx, a, t) \geq M(Ax, Sx, a, t)$ for all $x, a \in X$ and $t > 0$.

(ii) R-weakly commuting if there exists some $R > 0$ such that $M(ASx, SAx, a, t) \geq M(Ax, Sx, a, t/R)$ for all $x, a \in X$ and $t > 0$.

(iii) R-weakly commuting to type (A_g) provided there exists some real number R such that $M(AAx, BAx, a, t) \geq M(Ax, Bx, a, t/R)$ for each $x, a \in X$ and $t > 0$.

Definition 2.6 Let A and B be two selfmappings of a fuzzy 2-metric space $(X, M, *)$. We say that A and B satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = t$ for some $t \in X$.

3. Main Results

The following results provide the common fixed point theorems using the notion of R-weakly commutativity of type (A_g).

Theorem 3.1 Let A and B be pointwise R -weakly commuting selfmaps of type (A_g) of a fuzzy 2-metric space $(X, M, *)$ such that

(i) $A(X) \subset B(X)$,

(ii) $M(Ax, Ay, a, ht) \geq \min\{M(Bx, By, a, t), M(Ax, Bx, a, t), M(Ay, By, a, t), M(Ay, Bx, a, t), M(Ax, By, a, t)\}$ (1)

$0 \leq h < 1$, $t > 0$. If A and B satisfy the property (E.A) and the range of either of A or B is a complete subspace of X, then A and B have a unique common fixed point.

Proof Since A and B are satisfying the property (E.A), there exists a sequence $\{x_n\}$ in X such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = p$ for some p in X. Since $p \in AX$ and $AX \subset BX$, there exists some point u in X such that $p = Bu$ where $p = \lim_{n \to \infty} Bx_n$. If $Au \neq Bu$, the inequality.

$$M(Ax_n, Au, a, ht) \geq \min\{M(Bx_n, Bu, a, t), M(Ax_n, Bx_n, a, t), M(Au, Bu, a, t), M(Au, Bx_n, a, t), M(Ax_n, Bu, a, t)\}$$

on letting $n \to \infty$ yields

$$M(Bu, Au, a, ht) > \min\{M(Bu, Bu, a, t), M(Bu, Bu, a, t), M(Au, Bu, a, t), M(Au, Bu, a, t), M(Bu, Bu, a, t)\}$$
$$= M(Bu, Au, a, t)$$

Hence $Au = Bu$.

Since A and B are R-weakly commuting of type (A_g), there exists $R > 0$ such that

$$M(AAu, BAu, a, t) \geq M(Au, Bu, a, t/R) = 1,$$

that is, $AAu = BAu$ and $AAu = ABu = BAu = BBu$. If $Au \neq AAu$, using (ii), we get

$$M(Au, AAu, a, ht) \geq \min\{M(Bu, BAu, a, t), M(Au, Bu, a, t), M(AAu, BAu, a, t), M(AAu, BAu, a, t), M(Bu, AAu, a, t)\}$$

$$= M(Au, AAu, a, t),$$

a contradiction. Hence, $Au = AAu$ and $Au = AAu = ABu = BAu = BBu$. Hence Au is a common fixed point of A and B. The case when AX is a complete subspace of X is similar to the above case since $AX \subset BX$. Hence we have the theorem.

Example 3.2 Let $X =$ [2,20] and M be the usual metric on $(X, M, *)$.

Define $f, g: X \to X$ by

$$fx = 2 \text{ if } x = 2 \text{ or } x > 5, fx = 6, \text{ if } 2 < x \leq 5$$

$$gx = 2 \text{ if } x = 2, gx = x + 4, \text{ if } 2 < x \leq 5$$

$$gx = \frac{4x+10}{15} \text{ if } x > 5.$$

Define $M(x, y, t) = \frac{t}{t + d(x, y, z)}$ for all $x, y \in X$ and $t > 0$.

Where

$$d(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}.$$

Here

$fX = \{2\} \cup \{6\}$

$gX = [2,6] \cup \{7\} \cup \{8\} \cup \{9\}$

which implies $fX \subset gX$.

Therefore f and g satisfy all the conditions of the above theorem which include of R-weakly commuting of type (A_g) and $(E.A)$ property and also $x = 2$ is the unique common fixed point.

The following result shows that the common fixed point exists at point of discontinuity in noncampatible maps with relaxing the $(E.A)$ property condition.

Theorem 3.3 Let A and B be non compatible point wise R-weakly commuting selfmaps of type (A_g) of a fuzzy 2-metric space $(X, M *)$ such that

(i) $AX \subset BX$

(ii) $M(Ax, Ay, a, ht) \geq \min\{M(Bx, By, a, t), M(Ax, Bx, a, t), M(Ay, By, a, t), M(Ay, Bx, a, t), M(Ax, By, a, t)\}$,

$0 \le h < 1, t > 0$. If the range of A or B is a complete subspace of X, then A and B have a unique common fixed point and the fixed point is the point of discontinuity.

Proof : Since A and B are non-compatible maps, there exists a sequence $\{x_n\}$ in X such that

$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = p$ for some $p \in X$.

But either $\lim_{n \to \infty} M(ABx_n, BAx_n, a, t) \neq 1$ or the limit does not exist.

Since $p \in AX$ and $AX \subset BX$, there exists some point u in X such that $p = Bu$ where $p = \lim_{n \to \infty} Bx_n$. If $Au \neq Bu$, the in equality

$$M(Ax_n, Au, a, ht) \ge \min\{M(Bx_n, Bu, a, t), M(Ax_n, Bx_n, a, t), M(Au, Bu, a, t), M(Au, Bx_n, a, t), M(Ax_n, Bu, a, t)\}$$

on letting $n \to \infty$ yields

$$M(Bu, Au, a, ht) \ge M(Au, Bu, a, t).$$

Hence $Au = Bu$.

Since A and B are R-weakly commuting of type (A_g), there exists $R > 0$ such that

$M(AAu, BAu, a, ht) \ge M(Au, Bu, a, t/R) = 1$, that is, $Au = BAu$ and $AAu = ABu = BAu = BBu$. If $Au \neq AAu$, using (ii)we get

$$M(Au, AAu, a, ht) \ge \min\{M(Bu, BAu, a, t), M(Au, Bu, a, t), M(AAu, BAu, a, t), M(AAu, BAu, a, t), M(Bu, AAu, a, t)\}$$
$$= M(Au, AAu, a, t),$$

a contradiction. Hence $Au = AAu$ and $Au = AAu = ABu = BAu = BBu$. Hence Au is a common fixed point of A and B. The case when AX is a complete subspace of X is similar to the above case since $AX \subset BX$.

Now we have to show that A and B are discontinuous at the common fixed point $p = Au = Bu$. If possible suppose A is continuous. Then there exists sequence $\{x_n\}$ such that we get $\lim_{n \to \infty} AAx_n = Ap = p$. By R-weakly commuting of type (A_g) implies that

$M(AAx_n, BAx_n, a, t) \ge M(Ax_n, Bx_n, a, t/R)$ =1

which on letting $n \to \infty$ this yields

$\lim_{n \to \infty} BAx_n = Ap = p$. This, in turn, yields $\lim_{n \to \infty} M(ABx_n, BAx_n, a, t) = 1$.

This contradicts the fact that $\lim_{n \to \infty} M(ABx_n, BAx_n, a, t)$ is either nonzero or nonexistent for the sequence

$\{x_n\}$ of (1). Hence A is discontinuous at the fixed point. Next, suppose that B is continuous. Then for the sequence $\{x_n\}$ of (1), we get $\lim_{n\to\infty} BAx_n = Bp = p$ and $\lim_{n\to\infty} BBx_n = Bp = p$. In view of these limits, the inequality

$$M(Ax_n, ABx_n, a, ht) \geq \min\{M(Bx_n, BBx_n, a, t),$$
$$M(Ax_n, BBx_n, a, t), M(ABx_n, BBx_n, a, t),$$
$$M(ABx_n, Bx_n, a, t), M(Ax_n, BBx_n, a, t)\}$$
$$= M(Ax_n, ABx_n, a, t)$$

yields a contradiction unless

$\lim_{n\to\infty} ABx_n = Ap = Bp$. But

$\lim_{n\to\infty} ABx_n = Bp$ and $\lim_{n\to\infty} BAx_n = Bp$

contradicts the fact that

$$\lim_{n\to\infty} M(ABx_n, BAx_n, a, t)$$

is either nonzero or nonexistent. Thus both A and B are discontinuous at their common fixed point. Hence we have the theorem.

Example 3.4

Let $Let\ X = [2,20]$ *and* M be the usual fuzzy metric on $(X, M, *)$.

Define $f, g: X \to X$ by

$fx = 2\ if\ x = 2\ or\ x > 5, fx = 6, if\ 2 < x \leq 5$

$gx = 2\ if\ x = 2, gx = 8\ if\ 2 < x \leq 5$

$$gx = \frac{4x+10}{15}\ if\ x > 5.$$

Define $M(x,y,t) = \frac{t}{t+d(x,y,z)}\ for\ all\ x,y \in X$

$and\ t > 0.$

where

$d(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}$.

clearly $fX \subset gX$

$Since\ fX = \{2\} \cup \{6\}$

$gX = [2,6] \cup \{8\}$

Further f and g satisfy the condition of pointwise R-weakly commuting of type (A_g) and x = 2 is the unique common fixed point of f and g.

Consider the Sequence $x_n = \{5 + \frac{1}{n} / n > 1\}$.

Then $\lim_{n\to\infty} fx_n = 2$ and $\lim_{n\to\infty} gx_n = 2$, $\lim_{n\to\infty} fgx_n = 6$ and $\lim_{n\to\infty} gfx_n = 2$

$$But \lim_{n\to\infty} M(fgx_n, gfx_n) \neq 1.$$

Therefore f and g are non compatible.

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