Volume : 5 | Issue : 3 | March 2016 ISSN - 2250-1991 | IF : 5.215 | IC Value : 77.65

Research Paper Mathematics

# On Ternary Quadratic Equation $7x^2-3y^2=z^2$

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**ABSTRACT**

The Ternary Quadratic Diophantine equation given by $7x^2-3y^2=z^2$ is analyzed for its different patterns of non-zero integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

**KEYWORDS** Ternary, Quadratic, Integral solutions

**Introduction:**
The theory of Diophantine equations offers a rich variety of fascinating problems (1-5). For an extensive review of sizable literature and various problems, one may refer [6-20]. This communication concerns with yet another interesting ternary quadratic equation $7x^2-3y^2=z^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

**Notations Used:**
$t_{m,n}$-Polygonal number of rank n with size m.

$P_n^k$-Pentagonal number of rank n with size k.

Method of Analysis:

The Ternary Quadratic Diophantine Equation to be solved for its non-zero distinct integral solution is

$7x^2-3y^2=z^2$ (1)

**PATTERN I**
On substitution of linear transformations ($u\neq v\neq 0$)

$x=u+3v,\ y=u+7v,\ z=2z$ (2)

in (1) leads to $u^2=z^2+21v^2$ (3)

The corresponding solutions of (3) is the form

$z=a^2-21b^2$

$v=2ab$ (4)

$u=a^2+21b^2$

In view of (4), the solutions of (1) can be written as

$x=a^2+21b^2+6ab$

$y=a^2+21b^2+14ab$

$z=2a^2-42b^2$

A few interesting properties observed are as follows:

1. $z(a,1) + 2x(a,1)-8t_{3,a} \equiv 0 \pmod 8$

2. $z(1,b) - 2y(1,b) + t_{90,b}+t_{82,b} \equiv 0 \pmod{110}$

3. $x(a,2) + z(a,2) - 6t_{3,a} \equiv -84 \pmod 9$

4. $x(1,b) - t_{64,b} + t_{22,b} \equiv 1 \pmod{27}$

5. $x(1,b) + y(1,b) - t_{102,b} + t_{18,b} \equiv 2 \pmod{62}$

6. $x(3a,2) + 3y(3a,2) - 72t_{3,a} \equiv 84 \pmod{252}$

7. $z(A, A+1) + y(A,A+1) - t_{10,A} + t_{6,A} - 28t_{3,A} \equiv 0 \pmod 2$

8. $z(A,(A+1)(A+2)) + y(A,(A+1)(A+2)) - t_{12,A} + t_{8,A} - 84P_A^3 \equiv 0 \pmod 2$

9. $z(A,A(A+1)) + y(A,A(A+1)) - 20t_{3,A} + t_{18,A} - 28P_A^5 \equiv 0 \pmod{17}$

10. $z(a,1) + 2x(a,1) - t_{98,a} + t_{90,a} \equiv 0 \pmod{16}$

**PATTERN II**
The solution of (3) is obtained from

$z=21m^2-n^2$

$v=2mn$ (5)

$u=21m^2+n^2$

Substituting (5) in (2), the corresponding integral solutions of (1) are given by

$x=21m^2+n^2+6mn$

$y=21m^2+n^2+14mn$

$z=42m^2-2n^2$

A few interesting properties observed are as follows.

1. $z(m,4) - y(m,4) - t_{104,m} + t_{62,m} \equiv -48 \pmod{35}$

2. $2x(m,2) - y(m,2) - t_{62,m} + t_{20,m} \equiv 4 \pmod{17}$

3. $x(m,1) + z(m,1) - t_{102,m} - t_{28,m} \equiv -1 \pmod{67}$

4. $y(m,3) + 2x(m,3) - t_{148,m} + t_{22,m} \equiv 27 \pmod{141}$

5. $y(m,2) - t_{32,m} - t_{14,m} \equiv 4 \pmod{47}$

6. $x(1,n) + 3y(1,n) - t_{16,n} + t_{8,n} \equiv 32 \pmod{52}$

7. Each of the following expressions represents a Nasty number

a) z (m,m) +2x(m,m)

b) z (m,m) –x (m,m)

**PATTERN III**
Equation (3) can be written as

$z^2 + 21 v^2 = u^2 *1$ (6)

Assume that $u = a^2+21b^2$ (7)

Write 21 as 21 = (i sqaureroot of (21)) (-i squareroot of 21 )(8)

Define 1 = (2+i square root of 21)(2 - i square root of21) / 25 (9)

Use (7) and (8) in (6) and employing the method of factorization.

z+i square root of 21 v)=1/5[(a+isquare root of 21b)2 (2+1 square root of 21)] (10)

Equating the real and imaginary parts in (10), we obtain

$z= (a^2-21b^2-21ab)$ (11)

$u= (a^2-21b^2+4ab)$ (12)

Our interest is to obtain the integral solutions, so that the values of z and v are integers for suitable choices of the parameters 'a' and 'b'

Put a =5A, b=5B in (7), (11) and (12), we get

$z=10A^2-210B^2-210AB$

$v= 5A^2-105B^2+20AB$ (13)

$u= 25A^2+525B^2$

Substituting (13) in (2), the corresponding integral solutions of (1) are given by

$x= 40A^2+210B^2+60AB$

$y= 60A^2-210B^2+140AB$ (14)

$z= 20A^2-420B^2-420AB$

Thus equation (14) represents non-zero distinct integral solutions of (1) in two parameters.

A few interesting properties observed are as follows.

1) $x (2,B)-z (2,B)- t_{1402,B} + t_{142,B} \equiv 80 \pmod{1590}$

2) $y (A,B)+2z (A,3)- t_{152,A} – t_{52,A} \equiv -9450 \pmod{2002}$

3) $y (A,1)-x (A,1)- 40 t_{3,A} \equiv -420 \pmod{60}$

4) $x (A,3)-z (A,3)- t_{102,A} + t_{62,A} \equiv 1290 \pmod{1460}$

5) $y (1,B)+4x (1,B)- t_{1002,B} + t_{262,B} \equiv 220 \pmod{1008}$

6) $y (2A,5)-z (2A,5)- 320 t_{3,A} \equiv 5250 \pmod{5440}$

7) $x (B+1,B)-2z(B+1,B) - t_{2202,B} + t_{102,B} -1800 t_{3,B} \equiv 0 \pmod{1050}$

8) $x (B(B+1),B)-2z(B(B+1),B) - 2700 t_{3,B}+ t_{602,B} -1800 P_3^5 \equiv 0$ (mod 1649)

**PATTERN IV**
Define 1 = (10+i square root of 21)(10-isquare root of 21) / 121 (15)

The same procedure applied to find solution for PATTERN III is applied and obtained solution correspondingly for PATTERN IV.

$x=154A^2+1848B^2+660AB$

$y=198A^2+924B^2+1540AB$ (16)

$z= 220A^2-4620B^2-924AB$

Thus equation (16) represents non-zero distinct integral solutions of (1) in two parameters.

A few interesting properties observed are as follows.

1. $x(A,1)+z(A,1)-t_{762,A}+t_{14,A} \equiv -2772 \pmod{110}$

2. $z(A,1)-y(A,1)-t_{50,A}+t_{6,A} \equiv -5544 \pmod{2442}$

3. $x(3A,4)+y(3A,4)-t_{6402,A}+t_{66,A} \equiv 14784 \pmod{29568}$

4. $y(5,2B)-2z(5,2B)-t_{100002,B}+t_{18690,B} \equiv -6050 \pmod{29536}$

5. $z (A,A(A+1) + 5y (A,A(A+1) – 2600 t_{3,A}+t_{182,A}-13552P_A^5 \equiv 0$ (mod 1389)

6. $z (A,(A+1) (A+2)) +5y(A,(A+1) (A+2)) – 2430 t_{3,A}+t_{12,A}-40656P_A^5 \equiv 0 \pmod{1219}$

7. $z(A,A+1)+5y(A,A+1) - t_{2002,A}-t_{422,A}-13552 t_{3,A} \equiv 0 \pmod{1208}$

8. $2y (A, (A+1) (A+2)) –x (A,(A+1) (A+2)) – 500 t_{3,A}+t_{18,A}+14520 P_A^3 \equiv 0 \pmod{257}$

9. $2y (A, A(A+1))–x (A,A(A+1)) – 684 t_{3,A}+t_{202,A}+4840 P_A^5 \equiv 0$ (mod 351)

10. $2y (A,A+1)-x(A,A+1)-t_{492,A}+t_{8,A}+4840t_{3,A} \equiv 0 \pmod{242}$

**Conclusion:**
To conclude, one may search for other patterns of solutions and their corresponding properties.

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