Research Paper Mathematics

On the Ternary Quadratic Equation $x^2+3xy+y^2=z^4$

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ABSTRACT

The non-trivial integral solutions of the ternary quadratic equation $x^2+3xy+y^2=z^4$ are obtained. Some interesting relations among the solutions are presented.

KEYWORDS ternary quadratic, integral solutions.

Introduction

Ternary quadratic equations are rich in variety. For an extensive review of sizable literature and various problems; one may refer [1-5]. In this communication, we consider yet another interesting ternary quadratic equation $x^2+3xy+y^2=z^4$ and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions are presented.

Method of Analysis:

The equation to be solved is $x^2+3xy+y^2=z^4$ (1)

Pattern: I

Setting

$$\left.\begin{aligned} x&=u+v\\ y&=u-v\end{aligned}\right\} \quad (2)$$

The equation (1) simplifies to

$$5u^2-v^2=z^4 \quad (3)$$

Equation (3) also takes the form

$$4u^2-v^2=z^4-u^2 \quad (4)$$

The solutions satisfying (4) are given by

$$\left.\begin{aligned} u&=p^2+q^2\\ v&=2(p^2+pq-q^2)\\ z^2&=q^2+4pq-p^2\end{aligned}\right\} \quad (5)$$

In view of (5), the integral solutions of (1) are found to be

$$\left.\begin{aligned} x&=3p^2+2pq-q^2\\ y&=3q^2-2pq-p^2\\ z^2&=q^2+4pq-p^2\end{aligned}\right\} \quad (6)$$

Assume that

Equation (6) also takes the form

$$q^2+4pq-p^2=\alpha^2 \quad (7)$$

$$(q+2p)^2=5p^2+\alpha^2 \quad (8)$$

The solutions of (8) are represented in the form

$$\left.\begin{aligned} p&=2rs\\ \alpha&=5r^2-s^2\\ q&=5r^2+s^2-4rs\end{aligned}\right\} \quad (9)$$

In view of (9), equation (6) is seen to be

$$\left.\begin{aligned} x&=60r^3s+12rs^3-20r^2s^2-(5r^2+s^2)^2\\ y&=75r^4+3s^4+90r^2s^2-140r^3s-28rs^3\\ z&=5r^2-s^2\end{aligned}\right\} \quad (10)$$

Observations:

1. When $r=4s$,
   (i) $2(y+3x)$ is a perfect number.
   (ii) $\frac{y+3x}{2}$ is represented by a quadratic number.
   (iii) $3(y+3x)$ is a nasty number.
2. When $s=2r$,
   (i) $(y+3x)$ is a perfect square.
   (ii) $6(y+3x)$ is a nasty number.
   (iii) $9(y+3x)$ is a quadratic number.
3. $z^2+x\equiv 0(\mathrm{mod}\,4)$
4. For the following choices of r and s namely
   (i) $r=p^2+q^2, s=2(p^2+pq-q^2)$
   (ii) $r=P^2+Q^2, s=2(Q^2-PQ-P^2)$ the value of z in each case is a perfect square.

For the sake of simplicity and clear understanding a few numerical examples are given in Table 1.a. below:

Table: 1.a.

r	s	x	y	z
1	2	55	−21	1
1	3	128	−48	−4
2	4	880	−336	4
3	5	3200	−1200	20
2	3	527	−189	11

Pattern: II

Using completion of square, the equation (1) reduces to

$$(2x+3y)^2-(2z^2)^2=5y^2 \qquad (11)$$

Choose two non-zero integers p and q such that

$$\left.\begin{aligned} p(2x+3y-2z^2)&=qy \\ q(2x+3y+2z^2)&=5py \end{aligned}\right\} \qquad (12)$$

Solving by the method of cross multiplication, another choice of solution of (1) is represented in the form as

$$\left.\begin{aligned} x&=(10p^2-12pq+2q^2)t \\ y&=8pqt \\ z&=(10p^2-2q^2)t \end{aligned}\right\} \qquad (13)$$

Observations

1. When $t=\alpha^2$, each of the expressions $3(2x+3y-2z^2)$ and $15(2x+3y+2z^2)$ is a nasty number.
2. When $t=3\alpha^2$, $(2x-3y-2z^2)$ is a nasty number.
3. When $t=15\alpha^2$, $(2x+3y+2z^2)$ is a nasty number.
4. x is a perfect square for the choices of p and q given by $p=2R^2-S^2, q=10R^2-S^2$.
5. $z+x$ is a perfect square for the choices of p and q given by $p=36R^2, q=60R^2-3S^2$.

Reference

1. L.E. Dickson, History of Theory of numbers, Vol.2, Chelsea Publishing Company, New York, 1952.
2. L.J. Mordell, Diophantine Equations, Academic press, London, 1969.
3. Andre Weil, Number Theory: An approach through history: from hammurapi to legendre / Andre weil: Boston (Birkahasuserboston, 1983.
4. Nigel P. Smart, The algorithmic Resolutions of Diophantine equations, Cambridge university press, 1999.
5. Smith D.E History of mathematics vol.I and II, Dover publications, New York 1953.