Research Paper Mathematics

Quadratic Diophantine Equation With Four Unknowns $(x^2-y^2)^2+(z^2-8)^2=w^4$.

R.Anbuselvi Department of Mathematics, A.D.M. College for Women (Autonomous), Nagapattinam – 600 001, Tamil Nadu, India.

N.Ahila Department of Mathematics, Thiru.Vi.Ka.Govt. Arts College, Tiruvarur- 610003, Tamil Nadu, India.

ABSTRACT

The non-trivial integral solution of the quadratic Diophantine equation with four unknowns $(x^2-y^2)^2+(z^2-8)^2=w^4$. is obtained. A few interesting relations among the solutions are presented.

KEYWORDS quadratic, integral solutions.

Introduction

The Ternary Quadratic Diophantine Equation offers an unlimited field for research because of their variety. For an extensive review of various problems, one may refer [1-5].This communication concerns with yet another interesting ternary quadratic equation with four unknowns $(x^2-y^2)^2+(z^2-8)^2=w^4$. for determining its infinitely many non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

Method of Analysis

The equation under consideration is

$$(x^2-y^2)^2+(z^2-8)^2=w^4 \quad (1)$$

Taking

$$x^2-y^2=2ab \quad (2)$$

and

$$z^2-8=a^2-b^2 \quad (3)$$

in the equation (1), it is written as

$$a^2+b^2=w^2 \quad (4)$$

Again, setting

$$a=r^2-1,\quad b=2r \quad (5)$$

the equation (4) simplifies to

$$w=r^2+1 \quad (6)$$

and the equation (3) reduces to

$$z=r^2-3 \quad (7)$$

Substitution of (5) into (2) yields

$$x^2-y^2=4r(r^2-1) \quad (8)$$

Three different possibilities of solving (8) are analyzed below:

CASE (1):

Let

$$x-y=4$$
$$x+y=r^3-r \quad (9)$$

Solving the system of equations (9), we have

$$x=\frac{1}{2}(r^3-r+4)$$
$$y=\frac{1}{2}(r^3-r-4) \quad (10)$$

Thus the equations (6), (7) and (10) represent the non-trivial integral solutions of (1).

A few numerical examples are given in table 1(a) below:

Table: 8 .1(a)

r	x	y	z	w
1	2	–2	–2	2
2	5	1	1	5
3	14	10	6	10
4	32	28	13	17
5	62	58	22	26
6	107	103	33	37

The following results are noticed from the table 1(a):

(1) Each of the expressions $z+3$ and $w-1$ is a perfect square.

(2) Each of the expressions $x-y$ and $w-z$ is identically equal to four.

(3) $x+y+r$ is a cube.

(4) $(x+y)^2=(w-1)(w-2)^2=(z+3)(w-2)^2=(z+2)^2(z+3)=(z+2)^2(w-1)$

(5) $4(x+y)^2=(z+w)^2(z+3)=(z+w)^2(w-1)$

Case (2):

In (8), take

$$\begin{aligned} x-y&=r-1 \\ x+y&=4r(r+1) \end{aligned} \qquad (11)$$

Solving the above two equations, we have

$$\begin{aligned} x&=\frac{1}{2}(4r^2+5r-1) \\ y&=\frac{1}{2}(4r^2+3r+1) \end{aligned} \qquad (12)$$

Note that x and y are integers only when $r=2\alpha+1,\ \alpha=0,1,2,....$

Hence, the integral solutions of (1) are found to be

$$\begin{aligned} x&=8\alpha^2+13\alpha+4 \\ y&=8\alpha^2+11\alpha+4 \\ z&=4\alpha^2+4\alpha-2 \\ w&=4\alpha^2+4\alpha+2 \end{aligned} \qquad (13)$$

A few numerical examples are given in table 1(b) below:

Table: 1(b)

α	x	y	z	w
0	4	4	–2	2
1	25	23	6	10
2	62	58	22	26
3	115	109	36	40
4	184	176	78	82
5	269	259	118	122

The following results are noticed from the table 8.1(b):

(1) Each of the expression $x+y-4(2\alpha+1)$ and $y+z+w-3\alpha$ is a perfect square.

(2) $x+y$ is written as a difference of two squares.

(3) $w-z$ is identically equal to 4.

(4) $\dfrac{w+z}{\alpha^2+\alpha}$ is a cube.

(5) $z+w=16T_\alpha$

(6) $z+w=2(x-y)(x-y+2)$

(7) $x+y+1$ is always a perfect square.

(8) $3x-5y+2z+2w=-8.$

(9) $x-3y+2z+2w=-8.$

Case (3):

Now, consider the choice

$$x-y=4r \qquad (14)$$
$$x+y=r^2-1$$

Solving the above two equations, we have

$$x=\frac{1}{2}(r^2+4r-1) \qquad (15)$$
$$y=\frac{1}{2}(r^2-4r-1)$$

Note that x and y are integers only when $r=2\beta+1, \beta=0,1,2,....$

Hence, the integral solutions of (1) are found to be

$$x=2\beta^2+6\beta+2$$
$$y=2\beta^2-2\beta-2 \qquad (16)$$
$$z=4\beta^2+4\beta-2$$
$$w=4\beta^2+4\beta+2$$

A few numerical examples are given in table 8.1(c) below:

Table: 8.1(c)

β	x	y	z	w
0	2	–2	–2	2
1	10	–2	6	10
2	22	2	22	26
3	38	10	46	50
4	58	18	78	82
5	82	38	118	122

The following results are noticed from the table 8.1(c):

(1) $x+y=8T_\beta$

(2) Each of the expressions $w-z$ and $x+y+1$ is always a perfect square.

(3) $\frac{x-y-4}{\beta}$ is identically equal to 8.

(4) $z+w=16T_\beta$

(5) Each of the expressions $\frac{z+w-x+y}{2}$, $x+y$ and $8(z+w)$ is written as the difference of two squares.

(6) $x+y-z\equiv 0 \pmod 2$

(7) $\frac{w+z}{\beta^2+\beta}$ is a cube.

(8) $x+z-10\beta, (\beta>0)$ is a nasty number.

References

1. L.E. Dickson, History of Theory of numbers, Vol.2, Chelsea Publishing Company, New York, 1952.
2. L.J. Mordell, Diophantine Equations, Academic press, London, 1969.
3. Andre Weil, Number Theory: An approach through history: from hammurapi to legendre / Andre weil: Boston (Birkahasuserboston, 1983.
4. Nigel P. Smart, The algorithmic Resolutions of Diophantine equations, Cambridge university press, 1999.
5. Smith D.E History of mathematics vol.I and II, Dover publications, New York 1953.