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# On a Certain Subclass of Starlike Functions


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**ABSTRACT**

The aim of this paper is to introduce the class $STS_s(\alpha)$ $(0 < \alpha \le 1)$ satisfying the condition. $\left|\arg\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)\right| < \alpha \pi/2$

We study neighbourhoods of this class and also prove a necessary and sufficient condition in terms of convolution for a function f to be STSs (α). Further more, it is shown that class STSs (α) is closed under convolution with function f which are convex univalent in E.

**KEYWORDS** Stongly Starlike, Hadamard Product, Neighbourhood

**1. Introduction :** Let $A$ be the class of functions analytic in the unit disk $E$ normalized by $f(0) = f'(0) - 1 = 0$ and let $S$ denote the class of univalent functions in $A$ Let $ST(\alpha)$ $(0 \le \alpha < 1)$ denote the class of functions in $A$ that are starlike of order $\alpha$, and let $CV$ denote the class of convex functions. Then we have the classical analytic characterizations.

(1.1) $$f \in \mathrm{ST}(\alpha) \Leftrightarrow \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in E$$

and

(1.2) $$f \in \mathrm{CV} \Leftrightarrow \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in E$$

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Any $f \in A$ has the Taylor's expansion $f(z) = z + a_2 z^2 + \ldots..$ in $E$. The convolution or Hadamard product of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined as $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$ . Clearly

$$f(z) * \frac{z}{(1-z)^2} = zf'(z) \quad \text{and} \quad f(z) * \frac{z}{1-z^2} = \frac{f(z) - f(-z)}{2}$$

Strongly starlike and strongly convex functions were introduced and discussed by D.A. Brannan and W.E. Kirwan [1] and also by Stankiewincz [ 4 ] and [ 5 ]

The notion of $\delta$ - neighbourhood was introduced by St. Ruscheweyh [ 2 ].

**Definition 1.1.** For $\delta \geq 0$ the δ - neighbourhood of $f(z) \in A$ is define by

$$(1.3) \quad N_{\delta}(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n : \sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \right\}$$

To prove our results we need the following lemma

**Lemma A.** [ 3 ] If $\phi$ is a convex univalent function with $\phi(0) = \phi'(0) - 1$ in the unit disk $E$ and $g$ is starlike univalent in $E$, then for each analytic function $F$ in $E$, the image of $E$ under $\dfrac{(\phi * Fg)(z)}{(\phi * g)(z)}$ is a subset of the convex hull of $F(E)$

Now we give the definition of *STSs* (α) as follows.

**Definition 1.2.** A function $f(z)$ is said to be in the class $STS_S(\alpha)$ $(0 < \alpha \leq 1)$

if all $z \in E$.

(1.4) $$\left| arg\left( \frac{2zf'(z)}{f(z)-f(-z)} \right) \right| < \alpha \pi/2$$

$f \in STS_S(\alpha)$ means that the image of $E$ under $w = \frac{2zf'(z)}{f(z)-f(-z)}$. lies in the

region $\Omega = |arg\ w| < \alpha\ \pi/2$, equivalently $\frac{2zf'(z)}{f(z)-f(-z)} \neq t\ e^{\pm i\alpha\ \pi/2}$,

$t \in R^+$

Now let us give a characterization for a function $f \in A$ to be in $STS_S(\alpha)$ by means of convolution.

**Definition 1. 3 :** Let $STS'_S(\alpha)$ be the class of all analytic functions defined in

$E$ by

(1.5) $$H(z) = \frac{1}{1-te^{\pm i\alpha\pi/2}}\left[ \frac{z}{(1-z)^2} - te^{\pm i\alpha\pi/2}\left( \frac{z}{1-z^2} \right) \right],\ t \in R^+.$$

**Theorem 1.1.** $f \in STS_S(\alpha)$ if and only if $\frac{(f*H)(z)}{z} \neq 0,\ z \in E$ and for all

$H(z) \in STS'_S(\alpha)$.

**Proof:** Let us assume that $\frac{(f*H)(z)}{z} \neq 0$ then for all $H(z) \in STS'_S(\alpha)$ and

$z \in E$, We have.

$$\frac{(f*H)(z)}{z} = \frac{1}{z\left[1-te^{\pm i\alpha\pi/2}\right]}\left[f(z)*\frac{z}{(1-z)^2} - \left(te^{\pm i\alpha\pi/2}\right)\left(f(z)*\frac{z}{1-z^2}\right)\right]$$

$$= \frac{1}{z\left[1-te^{\pm i\alpha\pi/2}\right]}\left[zf'(z) - te^{\pm i\alpha\pi/2}\left(\frac{f(z)-f(-z)}{2}\right)\right]; \neq 0,\ t \in R^+$$

Equivalently $\frac{2zf'(z)}{f(z)-f(-z)} \neq te^{\pm i\alpha\pi/2}$ . As $t \in R^+$, $te^{\pm i\alpha\pi/2}$ covers the straight

line

$arg\ \ w = \pm(\alpha\pi/2)$

At $z = 0$, $\frac{2zf'(z)}{f(z)-f(-z)} = 1$, hence $f \in STS_S(\alpha)$.

Conversely let $f \in STS_S(\alpha)$. Then $\frac{2zf'(z)}{f(z)-f(-z)} \neq te^{\pm i\alpha\pi/2}$

or equivalently

$$f(z)*\left[\frac{z}{(1-z)^2} - te^{\pm i\alpha\pi/2}\left(\frac{z}{1-z^2}\right)\right] \neq 0, \quad \text{for} \quad z \neq 0$$

Normalising the function within the brackets we get

$\frac{(f * H)(z)}{z} \neq 0$ in $E$ where $H(z)$ is the function defined in ( 1. 5 )

**Lemma 1.1.** Let $H(z) = z + \sum_{n=2}^{\infty} c_n z^n \in STS'_S(\alpha)$. Then $|c_n| \leq \frac{n}{\sin\left(\alpha \pi/2\right)}$

**Proof:** Let $H(z) \in STS'_S(\alpha$ Then for $t \in R^+$

$$H(z) = \frac{1}{1 - te^{\pm i\alpha\pi/2}} \left[ \frac{z}{(1-z)^2} - te^{\pm i\alpha\pi/2} \left( \frac{z}{1-z^2} \right) \right]$$

$$= \frac{1}{1 - te^{\pm i\alpha\pi/2}} \left[ (z + 2z^2 + \ldots) - \left(te^{\pm i\alpha\pi/2}\right)(z + z^3 + \ldots) \right]$$

$$= z + \sum_{n=2}^{\infty} c_n z^n$$

Then comparing the coefficients on either side we get

$$c_n = \begin{cases} \dfrac{n}{1 - te^{\pm i\alpha\pi/2}}, & \text{when n is even} \\ \dfrac{n - te^{\pm i\alpha\pi/2}}{1 - te^{\pm i\alpha\pi/2}}, & \text{when n is odd} \end{cases}$$

Hence when $n$ is even

$$|c_n|^2 = \left|\frac{n}{1-te^{\pm i\alpha\pi/2}}\right|^2 = \frac{n^2}{(1-t\,cos(\alpha\pi/2))^2 + t^2\,sin^2\,(\alpha\pi/2)}$$

$$\frac{n^2}{(1-2t\,cos(\alpha\pi/2))^2 + t^2} = 1 + \frac{n^2 - 1 + 2t\,cos(\alpha\pi/2) - t^2}{1-2t\ \cos(\alpha\pi/2) + t^2}$$

$$\le \max_t \left[ 1 + \frac{n^2 - 1}{1-2t\ \cos(\alpha\pi/2) + t^2} \right]\ (\text{since } t \ge 0)$$

$$\le 1 + \frac{n^2-1}{sin^2(\alpha\pi/2)} = \frac{n^2 - \cos^2(\alpha\pi/2)}{sin^2(\alpha\pi/2)}$$

Therefore $|c_n| \le \dfrac{n}{sin(\alpha\pi/2)}$.

When $n$ is odd,

$$|c_n|^2 = \left|\frac{n-te^{\pm i\alpha\pi/2}}{1-te^{\pm i\alpha\pi/2}}\right|^2 = \frac{(n-t\,cos(\alpha\pi/2))^2 + t^2\,sin^2(\alpha\pi/2)}{(1-t\,cos(\alpha\pi/2))^2 + t^2\,sin^2(\alpha\pi/2)}$$

$$= \frac{(n^2 - 2nt\,cos(\alpha\pi/2))^2 + t^2}{(1-2t\,cos(\alpha\pi/2))^2 + t^2} = 1 + \frac{(n^2 - 1 - 2t(n-1)cos(\alpha\pi/2))}{(1-2t\,cos(\alpha\pi/2)) + t^2}$$

$$\le \max_t \left[ 1 + \frac{n^2 - 1}{1 - 2t\ cos\ (\alpha\pi/2) + t^2} \right]$$

$$\leq 1+\frac{\left(n^{2}-1-2t(n-1)\cos(\alpha\pi/2)\right)}{\left(1-2t\cos(\alpha\pi/2)\right)+t^{2}}, \qquad \text{since } t\geq 0$$

$$\leq 1+\frac{n^{2}-1}{\sin^{2}(\alpha\pi/2)}=\frac{n^{2}-\cos^{2}(\alpha\pi/2)}{\sin^{2}(\alpha\pi/2)}$$

There fore $|c_n|\leq \dfrac{n}{\sin(\alpha\pi/2)}$.

**Lemma 1.2**. For $f \in$ A and for every $\varepsilon \in$ C such that $|\varepsilon| < \delta$, if

$F_\varepsilon(z) = \dfrac{f(z)+\varepsilon z}{1+\varepsilon} \in STS_s(\alpha)$.

Then for every $H \in STS'_S(\alpha), \left|\dfrac{(f*H)(z)}{z}\right| > \delta, z \in E.$

**Proof.** Since $F_\in(z) \in STSs\,(\alpha)$, by Theorem 1 $\dfrac{(F_\in * H)(z)}{z} \neq 0$.

That is

$$\frac{(f*H)(z)+\in z}{(1+\in)z}\neq 0 \quad \text{or} \qquad \frac{(f*H)(z)}{z}\neq -\varepsilon, \quad \text{that is} \left|\frac{(f*H)(z)}{z}\right|\geq\delta .$$

**Theorem1. 2.** For $f \in A$ and $\varepsilon \in C$, $|\varepsilon| < \delta < 1$, assume $F_\varepsilon(z) \in STS_S(\alpha)$.

Then

$N_\delta(f) \subset STS_S(\alpha)$ where $\delta' = \delta \sin(\alpha\pi/2)$.

**Proof:** Let $H \in STS'_s(\alpha)$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is in $N_{\delta'}(f)$ Then

$$\left|\frac{(g*H)(z)}{z}\right| = \left|\frac{(f*H)(z)}{z} + \frac{((g-f)*H)(z)}{z}\right|$$

$$\geq \left|\frac{(f*H)(z)}{z}\right| - \left|\frac{(g-f)(z)*H(z)}{z}\right|$$

$$\geq \delta - \left|\sum_{n=2}^{\infty} \frac{(b_n - a_n)c_n z^n}{z}\right| \text{ by lemma 1.2}$$

thus

$$\left|\frac{(g*H)(z)}{z}\right| \geq \delta - |z| \sum_{n=2}^{\infty} |c_n||b_n - a_n|$$

$$> \delta - \frac{1}{sin(\alpha\pi/2)} \sum_{n=2}^{\infty} n|b_n - a_n| \text{ by lemma 1.1}$$

$$> \delta - \frac{\delta'}{sin(\alpha\pi/2)} = 0 \quad \text{for } \delta' = \delta \, sin(\alpha\pi/2)$$

Thus $\frac{(g*H)(z)}{z} \neq 0$ in $E$ for all $H \in STS'_S(\alpha)$ which means by Theorem1. 2,

$g \in STS_S(\alpha)$; in other words

$N_{\delta \, sin(\alpha\pi/2)}(f) \subset STS_S(\alpha)$.

**Lemma 1.3**. If g $\in STS_s(\alpha)$ then $G(z)=\dfrac{g(z)-g(-z)}{2}\in STS(\alpha)\subset ST(\alpha)$.

**Proof.** Since $g\in STS_s(\alpha), \dfrac{2zg'(z)}{g(z)-g(-z)}\in \Omega$, Now

$$\frac{zG'(z)}{G(z)}=\frac{zg'(z)}{2G(z)}+\frac{(-z)g'(z)}{2G(-z)}$$

$$=\frac{\zeta_1}{2}+\frac{\zeta_2}{2} \text{ where } \zeta_1 \text{ and } \zeta_2\in\Omega$$

$$=\zeta_3$$

Since $\Omega$ is convex $\zeta_3 \in \Omega$ and hence $\dfrac{zG'(z)}{G(z)}\in\Omega$ It can be easily seen that

$STS(\alpha)\subset ST(\alpha)$.

Thus $G(z)\in STS(\alpha)\subset ST(\alpha)$.

**Theorem 1.3.** Let $f\in CV$ and $g\in STSs(\alpha)$. Then $(f * g)(z)\in STSs(\alpha)$.

**Proof.** Let $f(z)\in CV$, $g(z)\in STSs(\alpha)$, $G(z)=\dfrac{g(z)-g(-z)}{2}$ and $\Omega$ is a

convex domain. Since $g(z)\in STSs(\alpha)$, $G(z)=\dfrac{g(z)-g(-z)}{2}\in \text{ST}(\alpha)$, by

Lemma 1.3.

Hence by an application of Lemma A we get

$$\frac{z(f*g)'(z)}{(f*G)(z)} = \frac{(f*zg')(\mathrm{z})}{(f*G)(z)} = \frac{f*\frac{zg'(z)}{G(z)}G(z)}{(f*G)(z)} \subset \overline{\mathrm{C}_0}\left(\frac{zg'(z)}{G(z)}\right) \subset \Omega$$

Since $\Omega$ is convex and $g \in STS_s\ (\alpha)$. This proves that $(f * g)(z) \in STSs$ $(\alpha)$.

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