Original Research Paper Mathematics

A STUDY ON USAGE OF EULERIAN AND HAMILTONIAN IN COMPLETE LINE BIGRAPH

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ABSTRACT

There are many games and puzzles which can be analyzed by graph theoretic concepts. In fact, the two early discoveries which led to the existence of graphs arose from puzzles, namely, the Konigsberg Bridge Problem and Hamiltonian Game, and these puzzles also resulted in the special types of graphs, now called Eulerian graphs and Hamiltonian graphs. Due to the rich structure of these graphs, they wide use both in research and application. In recent years, graph theory has established itself as an important mathematical tool in a wide variety of subjects, ranging from operational research and chemistry to genetics and linguistics, and from electrical engineering and geography to sociology and architecture. At the same time it has also emerged as a worthwhile mathematical discipline in its own right. In view of this, there is a need for an inexpensive introductory text on the subject, suitable both for mathematicians taking courses in graph theory and also for non specialists wishing to learn the subject as quickly As possible. It is my hope that this book goes some way towards filling this need. The only prerequisites to reading it are a basic knowledge of elementary set theory and matrix theory, although a further knowledge of abstract algebra is needed for more difficult exercises.

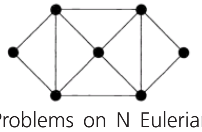
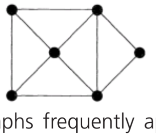
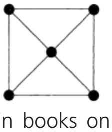
KEYWORDS graph theory, Eulerian graphs and Hamiltonian graphs etc.

INTRODUCTION

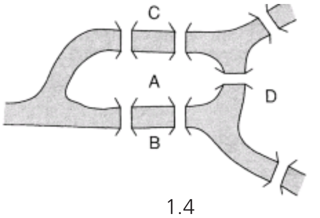
The origin of graph theory started with the problem of Koinsber bridge, in 1735. This problem lead to the concept of Eulerian Graph. Euler studied the problem of Koinsberg bridge and constructed a structure to solve the problem called Eulerian graph. In 1840, A.F Mobius gave the idea of complete graph and bipartite graph and Kuratowski proved that they are planar by means of recreational problems. The concept of tree, (a connected graph without cycles was implemented by Gustav Kirchhoff in 1845, and he employed graph theoretical ideas in the calculation of currents in electrical networks or circuits. A closed walk in a graph G containing all the edges of G is called an Euler line in G. A graph containing an Euler line is called an Euler graph. We knowthat a walk is always connected. Since the Euler line (which is a walk) contains all the edges of the graph, an Euler graph is connected except for any isolated vertices the graph may contain. As isolated vertices do not contribute anything to the understanding of an Euler graph, it is assumed now onwards that Euler graphs do not have any isolated vertices and are thus connected. A cycle passing through all the vertices of a graph is called a Hamiltonian cycle. A graph containing a Hamiltonian cycle is called a Hamiltonian graph. A path passing through all the vertices of a graph is called a Hamiltonian path and a graph containing a Hamiltonian path is said to be traceable. Examples of Hamiltonian graphs.

EULERIAN GRAPHS

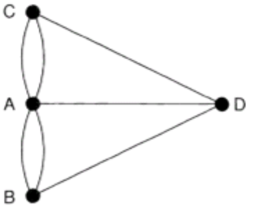
A connected graph G is Eulerian if there exists a closed trail containing every edge of G. Such a trail is an Eulerian trail. Note that this definition requires each edge to be traversed once and once only, A non-Eulerian graph G is semi-Eulerian if there exists a trail containing every edge of G. Figs 1.1, 1.2 and 1.3 show graphs that are Eulerian, semi-Eulerian and non-Eulerian, respectively.

Problems on N Eulerian graphs frequently appear in books on recreational mathematics. A typical problem might ask whether a given diagram can be drawn without lifting one's pencil from the paper and without repeating any lines. The name 'Eulerian' arises from the fact that Euler was the first person to solve the famous Konigsberg bridges problem which asks whether you can cross each of the seven bridges in Fig. 1.4 exactly once and return to your starting point. This is equivalent to asking whether the graph in Fig. 1.5 has an Eulerian trail. A translation of Euler's paper, and a discussion of various related topics, may be found in Biggs, Lloyd and Wilson.



1.4



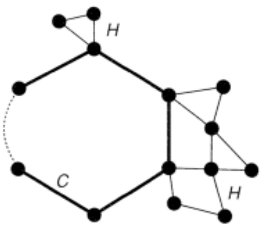
1.5

One question that immediately arises is 'can one find necessary and sufficient conditions for a graph to be Eulerian?' Before answering this question in Theorem 1.2, we prove a simple lemma. Proof. If G has any loops or multiple edges, the result is trivial. We can therefore sup pose that G is a simple graph. Let v be any vertex of G. We construct a walk v —> v\ —> v2 —> " * * inductively by choosing v\ to be any vertex adjacent to v and, for each / > 1, choosing vz-+1 to be any vertex adjacent to vz except v^; the existence of such a vertex is guaranteed by our hypothesis. Since G has only finitely many vertices, we must eventually choose a vertex that has been chosen before. If v# is the first such vertex, then that part of the walk lying between the two occurrences of v^ is the required cycle. //

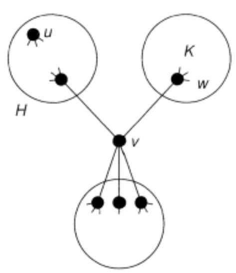
 ISSN - 2250-1991 | IF : 5.215 | IC Value : 79.96

Proof. => Suppose that P is an Eulerian trail of G. Whenever P passes through a vertex, there is a contribution of 2 towards the degree of that vertex. Since each edge occurs exactly once in P, each vertex must have even degree. <= The proof is by induction on the number of edges of G. Suppose that the degree of each vertex is even. Since G is connected, each vertex has degree at least 2 and so, by Lemma 1.1, G contains a cycle C. If C contains every edge of G, the proof is complete. If not, we remove from G the edges of C to form a new, possibly disconnected, graph H with fewer edges than G and in which each vertex still has even degree. By the induction hypothesis, each component of H has an Eulerian trail. Since each component of H has at least one vertex in common with C, by connectedness, we obtain the required Eulerian trail of G by following the edges of C until a non-isolated vertex of H is reached, tracing the Eulerian trail of the component of H that contains that vertex, and then continuing along the edges of C until we reach a vertex belonging to another component of H, and so on. The whole process terminates when we return to the initial vertex (see Fig. 1.6). //

This proof can easily be modified to prove the following two results. We omit the details.



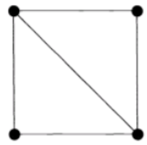
Note that, in a semi-Eulerian graph, any semi-Eulerian trail must have one vertex of odd degree as its initial vertex and the other as its final vertex. Note also that, by the handshaking lemma, a graph cannot have exactly one vertex of odd degree. We conclude our discussion of Eulerian graphs with an algorithm for constructing an Eulerian trail in a given Eulerian graph. The method is known as Fleury's algorithm. Proof. We show first that the construction can be carried out at each stage. Suppose that we have just reached a vertex v. If v ^ u, then the subgraph H that remains is connected and contains only two vertices of odd degree, u and v. To show that the construction can be carried out, we must show that the removal of the next edge does not disconnect H - or, equivalently, that v is incident with at most one bridge. But if this is not the case, then there exists a bridge vw such that the component KofH-vw containing w does not contain u (see Fig. 1.7). Since the vertex w has odd degree in K, some other vertex of K must also have odd degree, giving the required contradiction. If v = w, the proof is almost identical, as long as there are still edges incident with u.



1.7

HAMILTONIAN GRAPHS

In the previous section we discussed whether there exists a closed trail that includes every edge of a given connected graph G. A similar problem is to determine whether there exists a closed trail passing exactly once through each vertex of G. Note that such a trail must be a cycle, except when G is the graph N\. Such a cycle is a Hamiltonian cycle and G is a Hamiltonian graph. A non-Hamiltonian graph G is semi-Hamiltonian if there exists a path passing through every vertex. Figs 2.1, 2.2 and 2.3 show graphs that are Hamiltonian, semi-Hamiltonian and non-Hamiltonian, respectively.



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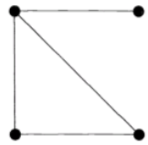

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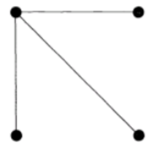
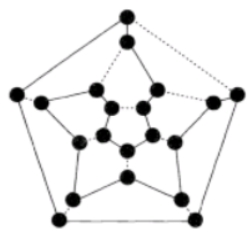


2.3

The name 'Hamiltonian cycle' arises from the fact that Sir William Hamilton investigated their existence in the dodecahedron graph, although a more general problem had been studied earlier by the Rev. T.P. Kirkman. Such a cycle is shown in Fig. 7.4, with heavy lines denoting its edges.



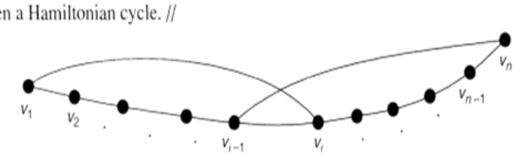
2.4

In Theorem 1.2 and Corollary 1.3 we obtained necessary and sufficient conditions for a connected graph to be Eulerian, and we may hope to obtain similar characterizations for Hamiltonian graphs. As it happens, the finding of such a characterization is one of the major unsolved problems of graph theory! In fact, little is known in general about Hamiltonian graphs. Most existing theorems have the form, 'if G has enough edges, then G is Hamiltonian'. Probably the most celebrated of these is due to G.A. Dirac, and known as Dirac's theorem. We deduce it from the following more general result of O. Ore.

Proof We assume the theorem false, and derive a contradiction. So let G be a non Hamiltonian graph with n vertices, satisfying the given condition on the vertex degrees. By adding extra edges if necessary, we may assume that G is 'only just' non Hamiltonian, in the sense that the addition of any further edge gives a Hamiltonian graph.

$$v_1 \to v_2 \to \cdots \to v_{i-1} \to v_n \to v_{n-1} \to \cdots \to v_{i+1} \to v_i \to v_1$$

is then a Hamiltonian cycle. //



2.5

CONCLUSION

Two concepts that are well-known and studied in Graph Theory are: Eulerian and Hamiltonian (di)cycles. An Eulerian (di)cycle in a (di)graph is a (di)cycle C such that an edge (resp. an arc) appears exactly once in C. A close notion is the Hamiltonian (di)cycle: where a vertex appears exactly once . A graph is Eulerian iff every vertex has an odd degree, and a digraph is Eulerian iff every vertex has equal indegree and outdegree. Therefore deciding if there is an Eulerian (di)cycle in a (di)graph G can be done in polynomial time; but deciding if there is a Hamiltonian (di)cycle is an NP-complete problem . A recent work generalizes the graph-theoretic concept of an Euler cycle to undirected hypergraphs. We now generalize the digraph-theoretic concept of an Eulerian dicycle to directed ones. We say that an Eulerian dicycle in a dihypergraph is a dicycleC such that a hyperarc appears exactly once in C. It is a natural generalization of an Eulerian dicycle in a digraph. We also define and study a generalization of Hamiltonian dicycles to directed hypergraphs.

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