Original Research Paper Mathematics

Numerical solution of Fifth order Runge-Kutta formula based on Contra - Harmonic Mean for the first order Fuzzy Initial Value Problems

R. Gethsi Sharmila Department of Mathematics, Bishop Heber College(Autonomous), Tiruchirappalli -17, Tamil Nadu, India.

E. C. Henry Amirtharaj Department of Mathematics, Bishop Heber College(Autonomous), Tiruchirappalli -17, Tamil Nadu, India.

ABSTRACT

In this paper, a numerical algorithm for solving fuzzy initial value problem based on Seikkala's derivative of fuzzy process by fifth order Runge-Kutta method based on Contra - Harmonic Mean(RK5CoM) is proposed. The algorithm is illustrated by solving a linear Fuzzy Initial Value Problem (FIVP) with triangular, trapezoidal, and parallelogram fuzzy numbers. The comparison is made between the classical fifth order Runge – Kutta Method and the proposed method. The results show that the proposed method gives good accuracy when solving the linear fuzzy initial value problems.

KEYWORDS Numerical solution, Fuzzy initial value problem, Fifth order Runge-Kutta method based on Contra - Harmonic mean, The Classical fifth order Runge – Kutta method, triangular, trapezoidal, parallelogram fuzzy numbers.

1. Introduction

Fuzzy set theory is a tool that makes possible to describe vague and uncertain notions. Fuzzy Differential Equation (FDE) models have wide range of applications in many branches of engineering and in the field of medicine. The concept of a fuzzy derivative was first introduced by Chang and Zadeh [5], later Dubois and Prade [6] defined the fuzzy derivative by using Zadeh's extension principle and then followed by Puri and Ralescu [18]. Fuzzy differential equations have been suggested as a way of modelling uncertain and incompletely specified systems and were studied by many researchers [9, 10, 11]. The existence of solutions of fuzzy differential equations has been studied by several authors . It is difficult to obtain exact solution for fuzzy differential equations and hence several numerical methods where proposed [13]. Abbasbandy and Allahviranloo [2] developed numerical algorithms for solving fuzzy differential equations based on Seikkala's derivative of fuzzy process [21]. Runge-Kutta method for fuzzy differential equation has been studied by many authors [1]. Murugesan et al. [14] compared fourth order RK methods based on variety of means and concluded that RKCeM works very well to solve system of IVPs and they also developed [15] a new embedded RK method based on AM and CeM.

Sanugi and Yaacob [20] developed a new fifth order five-stage Runge-Kutta method for initial value type problems in ODEs. Yaacob and Sanugi [22] studied and developed a fifth -order five-stage RK method based on Harmonic Mean. Ponalagusamy, Alphonse, and Chandru [17] gave new algorithm of fifth – order Heronian Mean Runge – Kutta method. Evans and Yaacub [8] developed a new fifth order weighted Runge – Kutta formula. Evans and Yaakub [7] proposed a fifth order Runge-Kutta RK(5, 5) method with error control In this paper, the new algorithms for fifth order Runge –Kutta method based on Contra - Harmonic Mean is developed and applied to solve fuzzy initial value problems with its initial value as triangular, trapezoidal and parallelogram fuzzy numbers . It is concluded from the example taken that the proposed methods RK5CoM works very well to solve fuzzy initial value problem.

The structure of the paper is organized as follows: In Section 2, some basic concepts of fuzzy set theory, fuzzy initial value problem, fifth order Runge-Kutta formula based on Contra - Harmonic Mean for solving Initial Value Problem is given. Fuzzy initial value problem is defined in Section 3. In section 4, numerical algorithm for solving the fuzzy initial value problems by the fifth order Runge-Kutta method based on the proposed method is discussed. The proposed algorithm is illustrated by an example in section 5 and the conclusion is in section 6.

2. Preliminaries

Definition 2.1. A fuzzy number is a fuzzy set u : □→ [0, 1]which satisfies

1. u is upper semi-continuous.
2. u(x) = 0 outside some interval [c, d],
3. there are real numbers a, b for which $c \le a \le b \le d$ such that

3.1. u(x) is monotonic increasing on [c, a],
3.2. u(x) is monotonic decreasing on [b,d], and
3.3. u(x) = 1, $a \le x \le b$.

Definition 2.2. A fuzzy number u in parametric form is a pair ((), ()), [0, 1],uururr=∈which satisfies the following requirements:()ur

1. is a bounded left continuous monotonic increasing function over [0, 1],()ur
2. is a bounded left continuous monotonic decreasing function over [0, 1], and
3. ≤ , $0 \le r \le 1$.

A crisp number α is simply represented by == α, $0 \le r \le 1$.

Definition 2.3.

A triangular fuzzy number *v*, is defined by three numbers where $a_1 < a_2 < a_3$ the graph of v(x) the membership function of the fuzzy number v, is a triangle with base on the interval $[a_1, a_3]$ and vertex at x = a2. And v is specified as $(a_1 / a_2 / a_3)$The membership function for the triangular fuzzy number $v = (a_1 / a_2 / a_3)$is defined as:

$$v(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \dfrac{x-a_3}{a_2-a_3}, & a_2 \le x \le a_3 \end{cases}$$

and one can write:

(1) $v > 0$ if $a_1 > 0$; (2) $v \ge 0$ if $a_1 \ge 0$;
(3) $v < 0$ if $a_3 < 0$; and (4) $v \le 0$ if $a_3 \le 0$.

Definition 2.4.

A trapezoidal fuzzy number u, is defined by four real numbers k < *l* < m < n where the base of the trapezoidal is the interval [k, n] and its vertices at x=*l*, x=m. Trapezoidal fuzzy number will be written as u = (k, *l*, m, n). The membership function for the trapezoidal fuzzy number u = (k, *l*, m, n) is defined as the following :

$$u(x)=\begin{cases}\frac{x-k}{l-k}, & k<x\le l\\ 1, & l<x<m\\ \frac{n-x}{n-m}, & m\le x<n\end{cases}$$

and one can have :
(1) u > 0 if k > 0 (2) u > 0 if l > 0;
(3) u > 0 if m > 0; and (4) u > 0 if n > 0.

Definition 2.5.
A parallelogram fuzzy number u is defined by four real numbers k< l< m< n, where the base of the parallelogram is the interval [k, n] and its vertices at x = l, x = m. Parallelogram fuzzy number will be written as u = (k, l, m, n). The membership function for the parallelogram fuzzy number u = (k, l, m, n) is defined as the following:

$$u(x)=\begin{cases}\frac{x-k}{l-k}, & k\le x\le l\\ 1, & l\le x\le m\\ \frac{x-n}{m-n}, & m\le x\le n\end{cases}$$

and one can have :
(1) u > 0 if k > 0 (2) u > 0 if l > 0;
(3) u > 0 if m > 0; and (4) u > 0 if n > 0.

Let E be the set of all upper semi continuous normal convex fuzzy numbers with bounded r–level intervals. It means that is $vE\in$then r-level set

$$[v]_r=\{s\setminus v(s)\ge r\},\ 0<r\le 1,$$

is a closed bounded interval which is denoted by

$$[v]_r=[v_1(r),\ v_2(r)].$$

Lemma 2.1. Let v, w ∈ E and s a scalar, then for r ∈ (0, 1]

[v + w]r = [v_1(r) + w_1(r), v_2(r) + w_2(r)],[v – w]r = [v_1(r) – w_1(r), v2(r) – w_2(r)],[v · w]r = [min{v_1(r) · w_1(r), v_1(r) · w_2(r), v_2(r) · w_1(r), v2(r) · w_2(r)},max{v_1(r) · w_1(r), v_1(r) · w_2(r), v_2(r) · w_1(r), v_2(r) ·w_2(r)}]. [sv]r = s[v]r.

The collection of all fuzzy numbers with addition and multiplication is denoted by E^1 and is a convex cone.

Definition 2.5. For arbitrary fuzzy numbers $u=(\underline{u}(r),\bar{u}(r))$, and $v=(\underline{v}(r),\bar{v}(r))$, the Quantity

$$D(u,\ v)=\sup_{0\le r\le 1}\left\{\max\left[|\underline{u}(r)-\underline{v}(r)|,\ |\bar{u}(r)-\bar{v}(r)|\right]\right\} \tag{2.1}$$

is the distance between u and v.

The function D (u, v) is a metric on E^1. This metric function is equivalent to the one used by Puri and Ralescu [17] and Kaleva [9].

Definition 2.6. A function $f:\mathbb{R}\to E^1$ is called a fuzzy function. If for arbitrary fixed $t_0\in\mathbb{R}$ and ∈>0, δ>0 such that

$$|t-t_0|<\delta\Rightarrow D(f(t),\ f(t_0))<\in \tag{2.2}$$

exists, f is said to be continuous.

Suppose that $y:I\to E^1$ is a fuzzy function. The parametric form of y (t) is represented by $[y(t)]_r=[y_1(t,\ r),\ y_2(t,\ r)]$, $t\in I,\ r\in(0,1]$, (2.3)
where I is a real interval. The Seikkala [20] derivative $y'(t)$ of a fuzzy function y (t) is defined by

$$[y'(t)]_r=[y_1'(t,\ r),\ y_2'(t,\ r)]\qquad t\in I,\ r\in(0,1], \tag{2.3}$$

provided that this equation defines a fuzzy number.

2.1. The fifth order Runge-Kutta formula based on Contra - Harmonic Mean of IVPs

Consider the initial value problem

$$\frac{dy}{dt}=f(t,y(t)),\ a\le t\le b$$
$$y(a)=\alpha \tag{2.5}$$

The basis of all Runge-Kutta method is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1}-y_n=\sum_{i=1}^{m}w_ik_i \tag{2.6}$$

where for i = 1, 2, …, m, wi's are constants and

$$k_i=h.f\left(t_n+c_ih,\ y_n+h\sum_{i=1}^{i-1}a_{ij}k_j\right) \tag{2.7}$$

Equations (2.7) is to be exact for powers of h through h^m, because it is to be coincident with Taylor series of order m. Therefore, the truncation error T_m, can be written as

$$T_m=\gamma_m h^{m+1}+O(h^{m+2}). \tag{2.8}$$

The fifth order Runge-Kutta formula based on Contra - Harmonic Mean for solving initial value problem of the form $y'=f(t,\ y)$ may be written as follows:

$$y_{n+1}=y_n+h\begin{bmatrix}-0.1773157366\left(\frac{k_1^2+k_2^2}{k_1+k_2}\right)\\+1.0254553152\left(\frac{k_2^2+k_3^2}{k_2+k_3}\right)\\-0.0779114700\left(\frac{k_3^2+k_4^2}{k_3+k_4}\right)\\+0.2297718914\left(\frac{k_4^2+k_5^2}{k_4+k_5}\right)\end{bmatrix} \tag{2.9}$$

where,

$$k_1=f(y_n)$$
$$k_2=f(y_n+0.1017275411h\ k_1)$$
$$k_3=f(y_n-0.5236574475\ h\ k_1+1.1653361910\ h$$
$$k_4=f(y_n+4.7450804540\ h\ k_1-4.2354437705h$$
$$k_5=f(y_n-0.5736403905\ h\ k_1+0.9301175162$$
$$+0.4667978567h\ k_3+0.1767250176\ h\ k_4) \tag{2.10}$$

The classical fifth order Runge-Kutta formula for solving initial value problem of the form $y'=f(t,\ y)$ may be written as follows:

$$y_{n+1}=y_n+\frac{1}{6}\ k_1+4k_4+k_5] \tag{2.11}$$

where,

$$k_1=h\,f(t_n,y_n)$$
$$k_2=h\,f(t_n+\frac{h}{3},\ y_n+\frac{1}{3}\ k_1)$$
$$k_3=h\,f(t_n+\frac{h}{3},y_n+\frac{1}{6}(k_1+k_2)$$
$$k_4=h\,f(t_n+\frac{h}{2},\ y_n+\frac{1}{8}(k_1+3\ k_3)$$
$$k_5=h\,f(t_n+h,\ y_n+\frac{1}{2}(k_1-3k_3)+2\ k_4) \tag{2.12}$$
$$a=t_0<t_1<\ldots<t_N=b\ \text{and}\ h=\frac{(b-a)}{N}=t_{i+1}-t_i \tag{2.13}$$

The local truncation error (LTE) of the method is given by the following:
RK5CoM:

$$LTE_{CoM} = \begin{bmatrix} 0.0132485733ff_y^5 \\ +0.0202501069f^2f_y^3f_{yy} \\ +0.0095106268f^3f_yf_{yy}^2 \\ -0.0022879188f^3f_y^2f_{yyy} \\ -0.0001379536\, f^4f_{yy}f_{yyy} \\ -0.0003448339f^4f_yf_{yyy} \\ -0.0000178190f^5f_{yyyyy} \end{bmatrix} h^6 + O(h^7) \quad (2.14)$$

3. Fuzzy Cauchy Problem

Consider the fuzzy initial value problem

$$y'(t) = f(t, y(t)); 0 \le t \le T$$
$$y(0) = y_0, \quad (3.1)$$

with the grid points

$$0 \le t_1 \le t_2 \le \ldots \le t_N = T \text{ and } h = \frac{(b-a)}{N} = t_{i+1} - t_i \quad (3.2)$$

where f is a continuous mapping from R+ × R into R and y0 ∈ E with r-level sets

$$[y_0]_r = [y_1(0; r),\ y_2(0; r)],\ r \in (0, 1],$$

The extension principle of Zadeh leads to the following definition of f (t, y) when y = y(t) is a fuzzy number

$$f(t, y)(s) = \sup\{y(\tau) \setminus s = f(t, \tau)\},\ s \in R$$

It follows that

$$[f(t, y)]_r = [f_1(t, y; r), f_2(t, y; r)],\ r \in (0, 1],$$

where

$$f_1(t, y; r) = \min\{f(t, u) \setminus u \in [y_1(r), y_2(r)]\}$$
$$f_2(t, y; r) = \max\{f(t, u) \setminus u \in [y_1(r), y_2(r)]\} \quad (3.3)$$

Theorem 3.1. Let f satisfy

$$\left|f(t, v) - f(t, \bar{v})\right| \le g(t, \left|v - \bar{v}\right|),\ t \ge 0,\ v,\ \bar{v} \in R.$$

where g : R+ × R+→R+ is a continuous mapping such that r → g(t, r) is non decreasing and the initial value problem

$$u'(t) = g(t, u(t)),\ u(0) = u_0. \quad (3.4)$$

has a solution on R+ for u0 > 0 and that u(t) ≡ 0 is the only solution of (3.4) for u0 = 0. Then the fuzzy initial value problem (3.1) has a unique solution.
Proof: see [21].

4. The fifth order RK methods based on Contra - Harmonic Mean for solving Fuzzy Initial Value Problems

We consider fuzzy initial value problem (3.1) with the grid points (3.2)
Let the exact solution [Y (t)] r = [Y_1(t; r), Y^2(t; r)] is approximated by some
[y(t)]r = [y_1(t; r), y_2(t; r)].
From (2.6), (2.7) we define

$$y_1(t_{n+1}; r) - y_1(t_n; r) = \sum_{i=1}^{5} w_i k_{i,1}(t_n, y(t_n; r)),$$
$$y_2(t_{n+1}; r) - y_2(t_n; r) = \sum_{i=1}^{5} w_i k_{i,2}(t_n, y(t_n; r)), \quad (4.1)$$

where the wi's are constants

$$[k_i(t, y(t; r))]_r =$$
$$[k_{i,1}(t, y(t; r), k_{i,2}(t, y(t; r))],\ i = 1, 2, 3, 4$$
$$k_{i,1}(t_n, y(t_n; r)) = h f(t_n + c_i h,\ y_1(t_n) + \sum_{j=1}^{i-1} a_{ij}\ k_{j,1}(t_n, y(t_n; r))),$$
$$k_{i,2}(t_n, y(t_n; r)) = h f(t_n + c_i h,\ y_2(t_n) + \sum_{j=1}^{i-1} a_{ij}\ k_{j,2}(t_n, y(t_n; r))), \quad (4.2)$$

and

$$\underline{k_1}(y_n(r)) = \min\{f(u) \setminus u \in [\underline{y_n}(r),\ \overline{y_n}(r)]\}$$
$$\overline{k_1}(y_n(r)) = \max\{f(u) \setminus u \in [\underline{y_n}(r),\ \overline{y_n}(r)]\}$$
$$\underline{k_2}(y_n(r)) = \min\{f(u) \setminus u \in [\underline{z_1}(y_n(r)),\ \overline{z_1}(y_n(r))]\}$$
$$\overline{k_2}(y_n(r)) = \max\{f(u) \setminus u \in [\underline{z_1}(y_n(r)),\ \overline{z_1}(y_n(r))]\}$$
$$\underline{k_3}(y_n(r)) = \min\{f(u) \setminus u \in [\underline{z_2}(y_n(r)),\ \overline{z_2}(y_n(r))]\}$$
$$\overline{k_3}(y_n(r)) = \max\{f(u) \setminus u \in [\underline{z_2}(y_n(r)),\ \overline{z_2}(y_n(r))]\}$$
$$\underline{k_4}(y_n(r)) = \min\{f(u) \setminus u \in [\underline{z_3}(y_n(r)),\ \overline{z_3}(y_n(r))]\}$$
$$\overline{k_4}(y_n(r)) = \max\{f(u) \setminus u \in [\underline{z_3}(y_n(r)),\ \overline{z_3}(y_n(r))]\}$$
$$\underline{k_5}(y_n(r)) = \min\{f(u) \setminus u \in [\underline{z_4}(y_n(r)),\ \overline{z_4}(y_n(r))]\}$$
$$\overline{k_5}(y_n(r)) = \max\{f(u) \setminus u \in [\underline{z_4}(y_n(r)),\ \overline{z_4}(y_n(r))]\} \quad (4.3)$$

where in the Runge-Kutta method of order five based on Contra - Harmonic Mean ,

$$\underline{z_1}(y_n(r)) = \underline{y_n}(r) + 0.1017275411\, h\, \underline{k_1}(y_n(r))$$
$$\overline{z_1}(y_n(r)) = \overline{y_n}(r) + 0.1017275411\, h\, \overline{k_1}(y_n(r))$$
$$\underline{z_2}(y_n(r)) = \underline{y_n}(r) - 0.5236574475\, h\, \underline{k_1}(y_n(r)) + 1.1653361910\, h\, \underline{k_2}(y_n(r))$$
$$\overline{z_2}(y_n(r)) = \overline{y_n}(r) - 0.5236574475\, h\, \overline{k_1}(y_n(r)) + 1.1653361910\, h\, \overline{k_2}(y_n(r))$$
$$\underline{z_3}(y_n(r)) = \underline{y_n}(r) + 4.7450804540\, h\, \underline{k_1}(y_n(r)) - 4.2354437705\, h\, \underline{k_2}(y_n(r)) + 0.0096366835\, h\, \underline{k_3}(y_n(r))$$
$$\overline{z_3}(y_n(r)) = \overline{y_n}(r) + 4.7450804540\, h\, \overline{k_1}(y_n(r)) - 4.2354437705\, h\, \overline{k_2}(y_n(r)) + 0.0096366835\, h\, \overline{k_3}(y_n(r))$$
$$\underline{z_4}(y_n(r)) = \underline{y_n}(r) - 0.5736403905\, h\, \underline{k_1}(y_n(r)) + 0.9301175162\, h\, \underline{k_2}(y_n(r)) + 0.4667978567\, h\, \underline{k_3}(y_n(r)) + 0.1767250176\, h\, \underline{k_4}(y_n(r))$$
$$\overline{z_4}(y_n(r)) = \overline{y_n}(r) - 0.5736403905\, h\, \overline{k_1}(y_n(r)) + 0.9301175162\, h\, \overline{k_2}(y_n(r)) + 0.4667978567\, h\, \overline{k_3}(y_n(r)) + 0.1767250176\, h\, \overline{k_4}(y_n(r)) \quad (4.4)$$

Define

$$F[y_n(r)] = h\begin{bmatrix} -0.1773157366\left(\dfrac{\underline{k_1}^2(y_n(r)) + \underline{k_2}^2(y_n(r))}{\underline{k_1}(y_n(r)) + \underline{k_2}(y_n(r))}\right) \\ +1.0254553152\left(\dfrac{\underline{k_2}^2(y_n(r)) + \underline{k_3}^2(y_n(r))}{\underline{k_2}(y_n(r)) + \underline{k_3}(y_n(r))}\right) \\ -0.0779114700\left(\dfrac{\underline{k_3}^2(y_n(r)) + \underline{k_4}^2(y_n(r))}{\underline{k_3}(y_n(r)) + \underline{k_4}(y_n(r))}\right) \\ +0.2297718914\left(\dfrac{\underline{k_4}^2(y_n(r)) + \underline{k_5}^2(y_n(r))}{\underline{k_4}(y_n(r)) + \underline{k_5}(y_n(r))}\right) \end{bmatrix}$$

$$G[y_n(r)] = h\begin{bmatrix} -0.1773157366\left(\dfrac{\overline{k_1}^2(y_n(r)) + \overline{k_2}^2(y_n(r))}{\overline{k_1}(y_n(r)) + \overline{k_2}(y_n(r))}\right) \\ +1.0254553152\left(\dfrac{\overline{k_2}^2(y_n(r)) + \overline{k_3}^2(y_n(r))}{\overline{k_2}(y_n(r)) + \overline{k_3}(y_n(r))}\right) \\ -0.0779114700\left(\dfrac{\overline{k_3}^2(y_n(r)) + \overline{k_4}^2(y_n(r))}{\overline{k_3}(y_n(r)) + \overline{k_4}(y_n(r))}\right) \\ +0.2297718914\left(\dfrac{\overline{k_4}^2(y_n(r)) + \overline{k_5}^2(y_n(r))}{\overline{k_4}(y_n(r)) + \overline{k_5}(y_n(r))}\right) \end{bmatrix} \quad (4.5)$$

The exact and approximate solutions at t_n, 0 ≤ n ≤ N are denoted by [Y (t_n)]r = [Y_1(t_n; r), Y_2(t_n; r)] and [y(t_n)]r = [y_1(t_n; r), y_2(t_n; r)] respectively. The solution is calculated by grid points at(2.13).

By (4.1) and (4.5), we have

$$\underline{Y_n}(r) \approx \underline{Y_n}(r) + F\left[Y_n(r)\right]$$
$$\overline{Y_n}(r) \approx \overline{Y_n}(r) + G\left[Y_n(r)\right] \quad (4.6)$$

We define

$$\underline{y}_n(r) \approx \underline{y}_n(r) + F\left[y_n(r)\right]$$
$$\overline{y}_n(r) \approx \overline{y}_n(r) + G\left[y_n(r)\right]$$
(4.7)

The lemmas given below will be applied to show convergence of these approximates in theorem 4.2. That is

$$\lim_{h \to 0} \underline{y}(t;\ r) = \underline{Y}(t;\ r)$$
$$\lim_{h \to 0} \overline{y}(t;\ r) = \overline{Y}(t;\ r)$$

Lemma 4.1 [13] Let the sequence of numbers $\{W_n\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \le A\,|W_n| + B,\ 0 \le n \le N-1,$$

for some given positive constants A and B. Then

$$|W_n| \le A^n\ |W_0| + B\ \frac{A^n - 1}{A-1},\ 0 \le n \le N.$$

Lemma 4.2 [13] Let the sequence of numbers $\{W_n\}_{n=0}^{N}, \{V_n\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \le |W_n| + A\ \max\{\,|W_n|,\ |V_n|\} + B,$$
$$|V_{n+1}| \le |V_n| + A\ \max\{\,|W_n|,\ |V_n|\} + B$$

for some given positive constants A and B, then denoting

$$U_n = |W_n| + |V_n|,\quad 0 \le n \le N.$$

Then

$$U_n \le \overline{A}^n U_0 + \overline{B}\frac{\overline{A}^n - 1}{A-1}, 0 \le \mathrm{n} \le \mathrm{N}.$$

where $\overline{A} = 1 + 2A$ and B = 2B.

Let F(u, v) and G(u, v) be obtained by substituting [yn(r)] = [u, v] in (4.5)

$$F[u,v] = h\begin{bmatrix} -0.1773157366\left(\dfrac{\underline{k}_1^2(u,v) + \underline{k}_2^2(u,v)}{\underline{k}_1(u,v) + \underline{k}_2(u,v)}\right) \\ +1.0254553152\left(\dfrac{\underline{k}_2^2(u,v) + \underline{k}_3^2(u,v)}{\underline{k}_2(u,v) + \underline{k}_3(u,v)}\right) \\ -0.0779114700\left(\dfrac{\underline{k}_3^2(u,v) + \underline{k}_4^2(u,v)}{\underline{k}_3(u,v) + \underline{k}_4(u,v)}\right) \\ +0.2297718914\left(\dfrac{\underline{k}_4^2(u,v) + \underline{k}_5^2(u,v)}{\underline{k}_4(u,v) + \underline{k}_5(u,v)}\right) \end{bmatrix}$$

$$G[u,v] = h\begin{bmatrix} -0.1773157366\left(\dfrac{\overline{k}_1^2(u,v) + \overline{k}_2^2(u,v)}{\overline{k}_1(u,v) + \overline{k}_2(u,v)}\right) \\ +1.0254553152\left(\dfrac{\overline{k}_2^2(u,v) + \overline{k}_3^2(u,v)}{\overline{k}_2(u,v) + \overline{k}_3(u,v)}\right) \\ -0.0779114700\left(\dfrac{\overline{k}_3^2(u,v) + \overline{k}_4^2(u,v)}{\overline{k}_3(u,v) + \overline{k}_4(u,v)}\right) \\ +0.2297718914\left(\dfrac{\overline{k}_4^2(u,v) + \overline{k}_5^2(u,v)}{\overline{k}_4(u,v) + \overline{k}_5(u,v)}\right) \end{bmatrix}$$

The domain of F and G is
$K = \{(t, u, v)\,|0 \le t \le T, -\infty < v < \infty, -\infty < u \le v\}$.

Theorem 4.1. Let F (t, u, v) and G (t, u, v) belong to C^5 (K) and let the partial derivatives of F and G be bounded over K. Then, for arbitrary fixed r, $0 \le r \le 1$, the approximate solutions (4.7) converge to the exact solutions Y_1 (r) and Y_2 (r) uniformly in t.

5. Numerical Examples

Example 5.1. Consider the fuzzy differential equation

$$\begin{cases} y'(t) = y(t),\ t \in [0,\ 1] \\ y(0) = (0.75 + 0.25r,\ 1.125 - 0.125r; \\ 0.8{+}0.125\mathrm{r},\ 1.1{+}0.1\mathrm{r}; \\ 1.1{+}0.1\mathrm{r}\sqrt{e},\ 1.5{+}0.1\mathrm{r}\sqrt{e}) \end{cases} \qquad (5.1)$$

The exact solution is given by

$$Y(t;r) = [(0.75+0.25r)e^t,\ (1.125-0.125r)e^t;$$
$$(0.8{+}0.125\mathrm{r})e^t,\ (1.1{+}0.1\mathrm{r})e^t;$$
$$\left(1.1{+}0.1\mathrm{r}\sqrt{e}\right)e^t,\ \left(1.5{+}0.1\mathrm{r}\sqrt{e}\right)e^t],$$
$$0 \le r \le 1.$$

At t=1 we get

$$Y(1;\ r) = [(0.75+0.25r)e^1,\ (1.125-0.125r)e^1;$$
$$(0.8{+}0.125\mathrm{r})e^1,\ (1.1{+}0.1\mathrm{r})e^1;$$
$$\left(1.1{+}0.1\mathrm{r}\sqrt{e}\right)e^1,\ \left(1.5{+}0.1\mathrm{r}\sqrt{e}\right)e^1],$$

The absolute error for the proposed method are compared with the classical fifth order RK method and are given in tables 5.1, 5.2, 5.3 respectively for the example 5.1 using tringular, trapezoidal and parallelogram fuzzy numbers when t=1 with the step size h = 0.01.

Table 5.1(for Triangular fuzzy number when h=0.01 and t=1)

Trape FN	Error in RK5CoM		Error in RK5	
r	y1	y2	y1	y2
0	4.40E-12	6.05E-12	3.02E-11	4.15E-11
0.1	4.47E-12	5.99E-12	3.07E-11	4.12E-11
0.2	4.54E-12	5.94E-12	3.11E-11	4.08E-11
0.3	4.60E-12	5.88E-12	3.16E-11	4.04E-11
0.4	4.68E-12	5.83E-12	3.21E-11	4.00E-11
0.5	4.74E-12	5.78E-12	3.26E-11	3.96E-11
0.6	4.81E-12	5.72E-12	3.30E-11	3.93E-11
0.7	4.88E-12	5.67E-12	3.35E-11	3.89E-11
0.8	4.95E-12	5.61E-12	3.40E-11	3.85E-11
0.9	5.02E-12	5.56E-12	3.45E-11	3.81E-11
1	5.09E-12	5.50E-12	3.49E-11	3.78E-11

Table 5.3 (For Parallelogram Fuzzy Number when h=0.01 and t=1)

Trape FN	Error in RK5CoM		Error in RK5	
r	y1	y2	y1	y2
0	6.05E-12	8.25E-12	4.15E-11	5.66E-11
0.1	6.14E-12	8.34E-12	4.22E-11	5.73E-11
0.2	6.23E-12	8.43E-12	4.28E-11	5.79E-11
0.3	6.32E-12	8.52E-12	4.34E-11	5.85E-11
0.4	6.41E-12	8.61E-12	4.40E-11	5.91E-11
0.5	6.50E-12	8.70E-12	4.46E-11	5.97E-11
0.6	6.59E-12	8.79E-12	4.53E-11	6.04E-11
0.7	6.69E-12	8.89E-12	4.59E-11	6.10E-11
0.8	6.78E-12	8.97E-12	4.65E-11	6.16E-11
0.9	6.86E-12	9.07E-12	4.71E-11	6.22E-11
1	6.96E-12	9.15E-12	4.78E-11	6.29E-11

The absolute error of RK5CoM is compared with the classical RK5 when h = 0.1 and t = 1 for the example 5.1 using triangular , trapezoidal and parallelogram fuzzy numbers and are represented in figures 5.1, 5.2, 5.3 respectively.

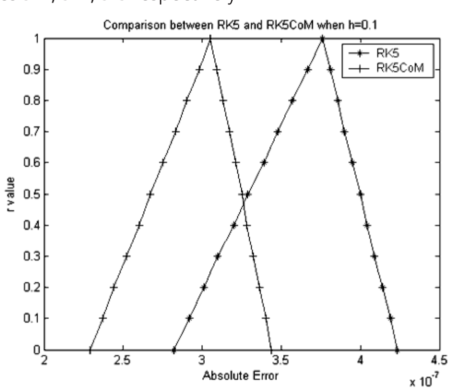

Figure 5.1 (t=1)

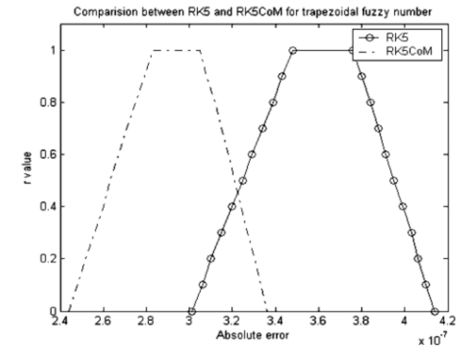

Figure 5.2 (when h=0.1 and t=1)

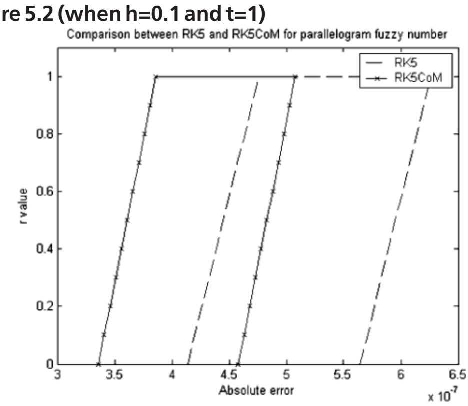

Figure 5.3 (h=0.1 and t=1)

6. Conclusions

The proposed Fifth Order Runge-Kutta method based on Contra - Harmonic Mean has been applied in this paper for finding the numerical solution of fuzzy differential equations. And it is compared with the classical fifth order Runge –Kutta method based on Arithmetic mean. From the tables 5.1, 5.2, 5.3 of example 5.1, the conclusion could be made for our proposed method that the fifth order Runge – Kutta method based on Contra - Harmonic Mean gives better solution than the classical fifth order Runge Kutta method. It can also be concluded that the triangular and trapezoidal fuzzy numbers works well for solving example 5.1 using the proposed method. The proposed fifth order Runge – Kutta methods based on Contra - Harmonic Mean gives good accuracy when the step size is taken to be minimum as h=0.01 and suits very well to solve fuzzy initial value problems with triangular, trapezoidal and parallelogram fuzzy numbers as its initial value.

Acknowledgements

This work has been supported by University Grants Commission, MRP – 5841/15,SERO, Hyderabad, India.

References

1. Abbasbandy. S, Allah Viranloo. T, Numerical solution of fuzzy differential equations by Runge-Kutta method, Nonlinear studies.11(2004)N0.1,117-129.
2. Abbasbandy. S, Allahviranloo. T, Numerical solution of fuzzy differential equations by Taylor method, Journal of Computational Methods and Applied Mathematics, 2(2002)113-124.
3. Buckley. J.J, Eslami. E, and Feuring. T, "Introductin to Fuzzy Logic and Fuzzy Sets", Physica-verlag, Heidelberg, Germany, 2001.
4. Butcher J.C., The Numerical Analysis of Ordinary Differential Equations Runge-Kutta and General Linear Methods, Wiley, New York, 1987.
5. Chang. S.L, Zadeh. L.A, On fuzzy mapping and control, IEEE Trans, Systems Man Cybernet. 2(1972)30-34.
6. Dubois. D, Prade. H, Towards fuzzy differential calculus part 3 : Differentiation, Fuzzy Sets and Systems, 8(1982)225-233.
7. Evans D. J. And Yaakub A.R. (2002) A Fifth Order Runge-Kutta Rk(5, 5) MethodWith Error Control, Intern. J. Computer Math., Vol. 79(11), pp. 1179–1185.
8. Evans, D.J. and Yaacub, A.R. (1996). A new fifth order weighted Runge – Kutta formula Intern. J. Computer Math, Vol.59, pp. 227-243.
9. Kaleva. O, Fuzzy differential equations, Fuzzy Sets and Systems, 24(1987)301-317.
10. Kaleva. O, The Cauchy problem for fuzzy differential equations, Fuzzy Sets and Systems, 35(1990)389 -396.
11. Lakshmikantham. V and Mohapatra. R, Theory of Fuzzy Differential Equations and Inclusions, Taylor and Francis, London, (2005).
12Lambert. J.D, Numerical Methods for ordinary differential systems, John Wiley &Sons, 1990.
13. Ma. M, Friedman. M, Kandel. A, Numerical solutions of fuzzy differential equations, Fuzzy Sets and Systems, 105(1999)133-138.
14. Murugesan. K, Paul Dhayabaran. D, Henry Amirtharaj. E.C and David J. Evans , A Comparison of extended Runge – Kutta formulae based on Variety of means to solve system of IVPs, Intern.J.Comp.Math.78 (2001), pp.225-252.
15. Murugesan. K, Paul Dhayabaran. D, Henry Amirtharaj. E.C and David J. Evans , A fourth order embedded Runge – Kutta RKACeM (4,4) method based on Arithmetic and Centroidal means with error conrol, Intern.J.Comp.Math.79(2) (2002), pp.247-269.
16. Palligkinis. S.ch., Papageorgiou. G, Famelis,I.TH., "Runge-Kutta methods for fuzzy differential equations", App.Math.Comp 209(2009),97-105.
17. Ponalagusamy. R, Alphonse. P. J. A, Chandru. M New algorithm of fifth – order Heronian Mean Runge – Kutta method, Recent Researches in Applied Mathematics and Informatics, pp. 67 -72.
18. Puri. M.L, Ralescu. D.A, Differentials of fuzzy functions, Journal of Mathematical Analysis and Applications, 91(1983)552-558.
19. Ralston. A, Rabinowitz. P, First Course in Numerical Analysis, McGraw Hill International Edition, (1978).
20. Sanugi, B.B and Yaacob, N (1995) A new fifth order five-stage Runge-Kutta method for initial value type problems in ODEs,Int. Jour. Comp. Math
21. Seikkala. S, On the fuzzy initial value problem, Fuzzy Sets and Systems, 24(1987)319-330.
22. Yaacob. N and Sanugi. B (1996) A fifth -order five-stage RK method based on Harmonic Mean, Differential Equations Theory, Numerics and applications, pp.381-389.