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FRACTIONAL ORDER QUADRATIC INTEGRAL EQUATION IN PARTIALLY ORDERED NORMED LINEAR SPACE

KEY WORDS: Partially Ordered Normed Linear Space, hybrid fixed point theorem, Fractional Order Quadratic Integral Equation.

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ABSTRACT

In this paper, we proved an existence theorem for a quadratic integral equation of fractional order in partially ordered normed linear space. Also provedthat the solution of this integral equation is locally attractive. Furthermore, we present an example.

I. Introduction:

The theory of integral equations is rapidly developing with the help of several tools of functional analysis, fixed point theoryand topology. In particular integral equations have many useful applications in the problem of the real world. The origin of the nonlinear integral equations lies in the work of Chandrasekhar [1] on radiative heat transfer in thermodynamics. Now a days it has become clear that such quadratic integral equations are applicable to theory of kinetic theory of gases, theory of neutron transport, queuing theory, radiative transfer, population dynamics and other. The previous methods for proving the existence results for such equations were much cumbersome, so this topic is not developed much during the initial stage of investigation. Many authors studied the existence the solution of nonlinear quadratic integral equations for several classes (see[6,10,12]) and their references, but the formulation of functional analytic methods, in particular, fixed point theory in partially ordered normed linear space, there is a considerable development of the nonlinear integral equations in recent years (see Dhage [3,4]) and the references therein.

The monotonic solution of a quadratic integral equation of fractional order

$$x(t) = a(t) + \frac{f(t,x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(s,x(s))}{(t-s)^{1-\alpha}} ds \tag{1.1}$$

has been studied in [10]. The main tool used in proof is the technique associated with the hausdorff measure of non-compactness.

In this paper, we investigate the existence the solution for a quadratic integral equation of fractional order (FQIE) in partially ordered normed linear space by using hybrid fixed point theorem due to B.C.Dhage.

$$x(t) = q(t,x(t))p(t) + \frac{f(t,x(t))}{\Gamma(\xi)} \int_0^t \frac{v(s,x(s))}{(t-s)^{1-\xi}} ds \tag{1.2}$$

for all $t \in \mathbb{R}_+$, and $\xi \in (0,1)$.

Where the function $p: \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, nonnegative,nondecreasing and bounded on $\mathbb{R}_+$. The function $q: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}, v: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous, nonnegative nondecreasing with respect to both variables t and x separately.

The article is organized as follows, in section 2 we give some preliminaries and fixed point theorem that will be used in subsequent part of paper. In section 3 we establish the main result and we provide an example to illustrate our result.

II. Preliminaries

In this section we give the definitions, notation, hypothesis and preliminary tools, which will be used in the sequel.

Definition2.1[6]: Let $\mathbb{E}$ be linear space or real vector. We introduce a partial order $\preccurlyeq$ in $\mathbb{E}$ as below. A relation in $\mathbb{E}$ is said to be partial order if it satisfies the following properties: let $a, b, c, d \in \mathbb{E}$ and $\lambda \in \mathbb{R}$.

1. Reflexivity:$a \preccurlyeq a \forall a \in \mathbb{E}$,

2. Antisymmetry: $a \preccurlyeq b$ and $b \preccurlyeq a$ implies$a = b$

3. Transitivity: $a \preccurlyeq b$ and $b \preccurlyeq c$ implies $a \preccurlyeq c$

4. Order linearity: $a \preccurlyeq b$ and $c \preccurlyeq d$ implies $a + c \preccurlyeq b + d$

The linear space $\mathbb{E}$ together with the partial order $\preccurlyeq$ becomes partially ordered vector orlinear space. Two elements x and y in a partially ordered linear space $\mathbb{E}$ are called comparable if either the relation $x \preccurlyeq y$ or $y \preccurlyeq x$ holds. We introduce a norm $\|.\|$ in a partially ordered linear space $\mathbb{E}$ so that$\mathbb{E}$ becomes partially ordered normed linear space. If $\mathbb{E}$ is complete with respect to the metric $\mathfrak{d}$ defined by the above norm, then it is called partially ordered complete normed linear space.

It is known that $\mathbb{E}$ is regular if $\{x_n\}$ is nondecreasing sequence in $\mathbb{E}$ such that $x_n \to x^*$as $n \to \infty$, then $x_n \preccurlyeq x^*$for all $n \epsilon \mathcal{N}$.

A some details of an ordered Banach space and operator theoretic techniques are given in the papers of Dhage [5], Lakshmikantham and Heikkilä [10] and Heikkiläand carl [9] and the also references therein.

Definition2.2 [5]: A mapping $G: \mathbb{E} \to \mathbb{E}$ is called isotone or monotone nondecreasing if it preserves the order relation$\preccurlyeq$, that is if$x \preccurlyeq y \Rightarrow Gx \preccurlyeq Gy \forall x, y \in \mathbb{E}$. Similarly G is called monotone non-increasing if$x \preccurlyeq y \Rightarrow Gx \succcurlyeq Gy$, $\forall x, y \in \mathbb{E}$. Finally G is called simply monotone if it is either monotonic nondecreasing or monotonic nonincreasing.

Definition2.3 [5]: An operator G on a normed linear space $\mathbb{E}$ into itself is called compact if G($\mathbb{E}$) is relatively compact subset of $\mathbb{E}$.
G is called totally bounded if for any bounded subset S of $\mathbb{E}$, G(S) is relatively compact subset of $\mathbb{E}$.If G is continuous and totally bounded, then it is called completely continuous on $\mathbb{E}$.

Definition2.4: A totally ordered subset $\mathfrak{C}$ of an ordered set $(\mathbb{E}, \preccurlyeq)$ is called chain in $\mathbb{E}$ or A totally ordered set is itself is a chain.

Definition2.5: A totally ordered set is a set and relation on the set that satisfy conditions of partial order and comparability condition. A relation $\preccurlyeq$ is a total order on a set $\mathbb{E}$ if the following properties hold.

1. Reflexivity:$a \preccurlyeq a \forall a \in \mathbb{E}$,

2. Antisymmetry: $a \preccurlyeq b$ and $b \preccurlyeq a$ implies $a = b$

3. Transitivity: $a \preccurlyeq b$ and $b \preccurlyeq c$ implies $a \preccurlyeq c$

4. Comparability: for any $a, b \in \mathbb{E}$ either $a \leq b$ or $a \geq b$

The first three axioms are of partial order and last one is trichotomy law defines a total order.

Definition2.6 [5]: A mapping $G: \mathbb{E} \to \mathbb{E}$ is called partially continuous at a point $b \in \mathbb{E}$ if for $\in > 0$ $\exists\, \delta > 0$ such that $\|Gx - Gb\| < \in$ whenever x is comparable to b and $\|x - b\| < \delta$.G is called partially continuous on $\mathbb{E}$, if it is partially continuous at every point of it. It is clear that if G is partially continuous on $\mathbb{E}$, then it is continuous on every chain $\mathfrak{C}$ contained in $\mathbb{E}$.

Definition 2.7[5]: An operator G on a partially normed linear space $\mathbb{E}$ into itself is called partially bounded if G($\mathfrak{C}$) is bounded for every chain $\mathfrak{C}$ in $\mathbb{E}$.

G is called uniformly partially bounded if all chains G($\mathfrak{C}$) in $\mathbb{E}$ are bounded by a unique constant. G is called partially compact if G($\mathfrak{C}$) is relatively compact subset of $\mathbb{E}$ for all totally ordered set or chains $\mathfrak{C}$ in $\mathbb{E}$.

G is called partially totally bounded if for any totally ordered and bounded subsets $\mathfrak{C}$ of $\mathbb{E}$, T($\mathfrak{C}$) is relatively compact subset of $\mathbb{E}$.

If G is partially continuous and partially totally bounded, then it is called partially completely continuous on $\mathbb{E}$.

Remark 2.1: Every compact mapping on a partially normed linear space is partially compact and every partially compact mapping is totally bounded, but the reverse implication is not true.

Again every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is partially continuous and partially totally bounded however the reverse implication may not true.

Definition2.8 [5]: The order relation $\preccurlyeq$ and the metric $\mathfrak{d}$ on a non empty set $\mathbb{E}$ are said to be compatible if $\{x_n\}$ is **monotone** that is monotone increasing or monotone decreasing sequences in $\mathbb{E}$ and if subsequence $\{x_{nk}\}$ of $\{x_n\}$ converges to x^* $\Longrightarrow$ the whole sequence $\{x_n\}$ converges to x^*.

Similarly, given a norm$\|.\|$and the partially normed linear space $(\mathbb{E}, \preccurlyeq, \|.\|)$ the order relation are said to be compatible if $\preccurlyeq$ and the metric $\mathfrak{d}$ defined through the norm $\|.\|$ are compatible.

The set of real numbers $\mathbb{R}$ with usual order relation $\preccurlyeq$ and the norm defined by absolute value function has this property. Similarly $\mathbb{R}^n$ is the finite dimensional Euclidean space with usual the standard norm and component wise order relation possesses the compatibility property. Similarly every partially compact subset of the space $C(\mathbb{R}_+, \mathbb{R})$ with usual order relation defined by $x \le y$ if and only if $x(t) \le y(t)$ for all $t \in \mathbb{R}_+$ with usual standard supremum norm $\|.\|$ defined by $\|x\| = sup_{t\in\mathbb{R}_+}|x(t)|$ are compatible.

Definition2.9 [6]: A mapping $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is called dominating function or in short $\mathcal{D}$-function if it is an upper semicontinuous and monotonic nondecreasing function satisfies $\varphi(0) = 0$.

Definition2.10 [5]: Let$(\mathbb{E}, \preccurlyeq, \|.\|)$ be a partially ordered normed linear space. A mapping $G: \mathbb{E} \to \mathbb{E}$ is called partially $\mathcal{D}$- lipschitz or partially nonlinear $\mathcal{D}$- lipschitz, if there exist an upper semi-continuous nondecreasing function $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$\|Gx - Gy\| \le \varphi\|x - y\|$(2.1)

For all comparable elements $x, y \in \mathbb{E}$ where $\varphi(0) = 0$, if $\varphi(r) = kr, k > 0$, then G is called partially lipschitz with lipschitz constant k.

If $k < 1$, G is called partially contraction with contraction constant k.

Finally G is called nonlinear $\mathcal{D}$-contractionif it is nonlinear $\mathcal{D}$- lipschitz with $\varphi(r) < r$ for $r > 0$.

Definition 2.11[5]: Let $(\mathbb{E}, \preccurlyeq, \|.\|)$ be a partially normed linear algebra. Denote $\mathbb{E}^+ = \{x \in \mathbb{E} | x \geq \theta,\}$ where θ is zero element of $\mathbb{E}$. and

$$\mathcal{K} = \{\mathbb{E}^+ \subset \mathbb{E} | uv \in \mathbb{E}^+, \forall u, v \in \mathbb{E}^+\}$$

The elements in set $\mathcal{K}$ are called the positive vectors in the normed linear algebra $\mathbb{E}$.

III. Existence Theory

Definition3.1[5]: Let $(\mathbb{E}, \preccurlyeq, \|.\|)$ be a partially normed linear algebra. Denote $\mathbb{E}^+ = \{x \in \mathbb{E} | x \geq \theta,\}$ where θ is zero element of $\mathbb{E}$. And$\mathcal{K} = \{\mathbb{E}^+ \subset \mathbb{E} | uv \in \mathbb{E}^+, \forall u, v \in \mathbb{E}^+\}$.

The elements in set $\mathcal{K}$ are called the positive vectors in the normed linear algebra $\mathbb{E}$.

Lemma3.1[5]: If $\mathcal{U}_1 \mathcal{U}_2 \mathcal{V}_1 \mathcal{V}_2 \in \mathcal{K}$ such that $\mathcal{U}_1 \leq \mathcal{V}_1$ and $\mathcal{U}_2 \leq \mathcal{V}_2$then$\mathcal{U}_1 \mathcal{U}_2 \leq \mathcal{V}_1 \mathcal{V}_2$.

Lemma3.2: An operator $G: \mathbb{E} \to \mathbb{E}$ is said to be positive if the range $R(G)$ of G is such that $R(G) \subset \mathcal{K}$ for any two chains $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in $\mathbb{E}$ denote

$$\mathfrak{C}_1 \mathfrak{C}_2 = \{x = c_1 c_2, c_1 \mathfrak{C}_1 and c_2 \mathfrak{C}_2\}.$$

Theorem3.1 [5]: Let S be nonempty, partially bounded and closed subset of a regular partially ordered complete algebra $(\mathbb{E}, \preccurlyeq, \|.\|)$ such that the order relation $\preccurlyeq$ and the norm $\|.\|$ are compatible in every compact chain $\mathfrak{C}$of S. Let $\mathbb{A}, \mathbb{B}: S \to \mathcal{K}$ and $\mathbb{C}: \mathbb{E} \to \mathbb{E}$ be two nondecreasing operators, such that

a) $\mathbb{A}$ and $\mathbb{C}$ are partially nonlinear $\mathcal{D}$- Lipschitz with $\mathcal{D}$-function $\varphi_{\mathbb{A}}$ and$\varphi_{\mathbb{C}}$ respectively**.**
b) $\mathbb{B}$ is partially continuous and compact
c) $\mathbb{A}x\mathbb{B}x + \mathbb{C}x \in S \forall x \in S$,
d) $M\varphi_{\mathbb{A}}(r) + \varphi_{\mathbb{C}}(r) < r, r > 0$, where $M = \|\mathbb{B}(S)\|$ and
e) There exist an element $x_0 \in S$ such that $x_0 \preccurlyeq \mathbb{A}x_0\mathbb{B}x_0 + \mathbb{C}x_0$ or $x_0 \succcurlyeq \mathbb{A}x_0\mathbb{B}x_0 + \mathbb{C}x_0$

Then the operator equation $\mathbb{A}x\mathbb{B}x + \mathbb{C}x = x$ has a solution x^* in S and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathbb{A}x_n\mathbb{B}x_n + \mathbb{C}x_n, n = 0,1,2, \dots \dots \dots$ Converges monotonically to x^*.

Now Fractional Quadratic Integral Equation (FQIE) (6.1.1) will be investigated under the following assumptions:

$\mathcal{H}$1) The function $p: \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, nonnegative, nondecreasing and bounded on $\mathbb{R}_+$ with bound $\mathbb{P} = sup_{t \geq 0} |p(t)|$.

$\mathcal{H}$ 2) The function $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous, nonnegative, nondecreasing with bound $\mathbb{F} = sup_{t \geq 0} |f(t, x)|$, there exist a nondecreasing and bounded function $\ell_{\mathbb{A}}(t): (0, \infty) \to \mathbb{R}_+$with bound $\|\ell_{\mathbb{A}}\|$such that

$$|f(t,x(t)) - f(t,y(t))| \leq \ell_{\mathbb{A}}(t)|x(t) - y(t)|,$$

$\forall t \in \mathbb{R}_+$ and $x \in \mathbb{R}_+$.

$\mathcal{H}$3) The function $q: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous, nonnegative, nondecreasing with bound $\mathbb{Q} = sup_{t\geq 0}|q(t,x(t))|$, there exist a nondecreasing and bounded function $\ell_{\mathbb{C}}(t): (0,\infty) \to \mathbb{R}_+$ with bound $\|\ell_{\mathbb{C}}\|$ such that $|q(t,x(t)) - q(t,y(t))| \leq \ell_{\mathbb{C}}(t)|x(t) - y(t)|$, $\forall t \in \mathbb{R}_+$.

$\mathcal{H}$4) The function $v: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous, nonnegative, nondecreasing satisfying $v(t,x(t)) \leq h(t)$ where $h: \mathbb{R}_+ \to \mathbb{R}$ for $\forall t \in \mathbb{R}_+$ and $x \in \mathbb{R}_+$.

$\mathcal{H}$5) The continuous function $\gamma: \mathbb{R}_+ \to \mathbb{R}_+$ defined by the formula

$\gamma(t) = \int_0^t \frac{h(t)}{(t-s)^{1-\xi}} ds.$

$\mathcal{H}_6$) There exist an element $u \in C(\mathbb{R}_+, \mathbb{R})$ such that

$$u(t) \leq q(t,u(t))p(t) + \frac{f(t,u(t))}{\Gamma(\xi)} \int_0^t \frac{v(s,u(s))}{(t-s)^{1-\xi}} ds$$

Remark 3.1: If the hypothesis ($\mathcal{H}$4) and ($\mathcal{H}$5) hold then there exist a constant $\mathcal{K}_1, \mathcal{K}_2 > 0$ such that $\mathcal{K}_1 = sup_{t\geq 0} \frac{v(t)}{\Gamma(\xi)} = sup_{t\geq 0} \frac{1}{\Gamma(\xi)} \int_0^t \frac{h(s)}{(t-s)^{1-\xi}} ds$ and

$\mathcal{K}_2 = \mathbb{P}\mathbb{Q} = sup_{t\geq 0}|p(t)||q(t,x(t))|$. And the function $q: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\gamma: \mathbb{R}_+ \to \mathbb{R}_+$ vanishes at infinity.

IV. Main Result:

Theorem4.1: Suppose that the assumption ($\mathcal{H}$1)-($\mathcal{H}$6), and $\mathcal{K}_1\|\ell_{\mathbb{A}}\| + \mathbb{P}\|\ell_{\mathbb{C}}\| < r$, where r is a positive real number. Then FQIE (1.2) has at least one solution $x = x(t)$ which belongs to partially ordered normed linear space and is nonnegative, nondecreasing on $\mathbb{R}_+$.

Proof: let $(\mathbb{E}, \preccurlyeq, \|.\|)$ be a regular partially ordered complete normed linear space and define a subset S of $(\mathbb{E}, \preccurlyeq, \|.\|)$ as $S = \{x \in (\mathbb{E}, \preccurlyeq, \|.\|) | \|x\| \leq \mathfrak{z}\}$.

Where $\mathfrak{z}$ satisfies the inequality $\mathbb{F}\mathcal{K}_1 + \mathcal{K}_2 \leq \mathfrak{z}$. Clearly S be nonempty, partially bounded and closed subset of a regular partially ordered complete algebra $(\mathbb{E}, \preccurlyeq, \|.\|)$.

Consider the chain $\mathfrak{C}$ in $S \subset \mathbb{E}$.
Define the operators $\mathbb{A}, \mathbb{B}: S \to S$ and $\mathbb{C}: \mathbb{E} \to \mathbb{E}$ such that

$$\mathbb{A} = f(t,x(t)) \quad (4.1)$$

$$\mathbb{B} = \frac{1}{\Gamma(\xi)} \int_0^t \frac{v(s,x(s))}{(t-s)^{1-\xi}} ds \quad (4.2)$$

$$\mathbb{C} = q(t,x(t))p(t) \quad (4.3)$$

Now the FQIE (1.2) is equivalent to the operator equation

$$x(t) = \mathbb{A}x(t)\mathbb{B}x(t) + \mathbb{C}x(t) \quad (4.4)$$

Now we will show that the operators satisfy conditions of theorem (3.1)

Step I: The operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are nondecreasingon $\mathbb{E}$. Let $x, y \in S$ such that $x \leq y$ then by hypothesis we obtain

$$\mathbb{A}x(t) = f\big(t, x(t)\big) \leq f\big(t, y(t)\big) \leq \mathbb{A}y(t),$$

$$\mathbb{B}x(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{v\big(s, x(s)\big)}{(t-s)^{1-\xi}} ds \leq \frac{1}{\Gamma(\xi)} \int_0^t \frac{v\big(s, y(s)\big)}{(t-s)^{1-\xi}} ds \leq \mathbb{B}y(t)$$

and $\mathbb{C}x(t) = q\big(t, x(t)\big)p(t) \leq q\big(t, y(t)\big)p(t) \leq \mathbb{C}x(t)$, for all $t \in \mathbb{R}_+$

Thus the operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are nondecreasing on S.

Step II: To show $\mathbb{A}$ and $\mathbb{C}$ are partially $\mathcal{D}$- Lipschitz with Lipschitz function $\ell_{\mathbb{A}}$ and $\ell_{\mathbb{C}}$ respectively.

$$|\mathbb{A}x(t) - \mathbb{A}y(t)| = \big|f\big(t, x(t)\big) - f\big(t, y(t)\big)\big|$$

$$\leq \ell_{\mathbb{A}}(t)|x(t) - y(t)|, \text{ for all } t \in \mathbb{R}_+$$

Taking supremum over $t \in \mathbb{R}_+$ in above inequality, we obtain

$$\|\mathbb{A}x - \mathbb{A}y\| \leq \|\ell_{\mathbb{A}}\|\|x - y\|, \forall x, y \in S$$

Hence $\mathbb{A}$ is partially nonlinear $\mathcal{D}$-Lipschitz on S with $\mathcal{D}$-Lipschitz constant $\|\ell_{\mathbb{A}}\|$.

Similarly $|\mathbb{C}x(t) - \mathbb{C}y(t)| = \big|q\big(t, x(t)\big)p(t) - q\big(t, y(t)\big)p(t)\big|$

$$= |p(t)|\big|q\big(t, x(t)\big) - q\big(t, y(t)\big)\big|$$

$$\leq \mathbb{P}\ell_{\mathbb{C}}(t)|x(t) - y(t)| \text{for all } t \in \mathbb{R}_+,$$

Taking supremum over $t \in \mathbb{R}_+$ in above inequality, we obtain

$$\|\mathbb{C}x - \mathbb{C}y\| \leq \mathbb{P}\|\ell_{\mathbb{C}}\|\|x - y\|, \forall x, y \in S.$$

Thus $\mathbb{A}$ and $\mathbb{C}$ are partially nonlinear $\mathcal{D}$-Lipschitz on S with $\mathcal{D}$-Lipschitz constant $\|\ell_{\mathbb{A}}\|$ and $\mathbb{P}\|\ell_{\mathbb{C}}\|$ respectively.

Step III: To show $\mathbb{B}$ is partially continuous and compact operator on S.

Firstly to show $\mathbb{B}$ is partially continuous operator on S.

Let $\{x_n\}$ be a sequence in a chain $\mathfrak{C}$ in $S \subset \mathbb{E}$ converging to a point x, then by dominated convergence theorem for all $t \in \mathbb{R}_+$, we obtain

Then, $\lim_{n\to\infty} \mathbb{B}x_n(t) = \lim_{n\to\infty} \left\{\frac{1}{\Gamma(\xi)} \int_0^t \frac{v\left(s, x_n(s)\right)}{(t-s)^{1-\xi}} ds\right\}$

$$= \frac{1}{\Gamma(\xi)} \int_0^t \frac{v\left(s, x(s)\right)}{(t-s)^{1-\xi}} ds$$

$$= \mathbb{B}x(t), \forall t \in \mathbb{R}_+$$

This shows that $\mathbb{B}x_n$ converges to $\mathbb{B}x$ point- wise on S.

Next to show that sequence $\{\mathbb{B}x_n\}$ is a equicontinuous sequence in S.

Let $t_1, t_2 \in \mathbb{R}_+$ be arbitrary with $t_1 < t_2$ then

$$|\mathbb{B}x_n(t_2) - \mathbb{B}x_n(t_1)| = \left|\frac{1}{\Gamma(\xi)}\int_0^{t_2}\frac{v(s,x_n(s))}{(t_2-s)^{1-\xi}}ds - \frac{1}{\Gamma(\xi)}\int_0^{t_1}\frac{v(s,x_n(s))}{(t_1-s)^{1-\xi}}ds\right|$$

$$\leq \frac{1}{\Gamma(\xi)}\left|\begin{matrix}\int_0^{t_2}(t_2-s)^{\xi-1}v(s,x_n(s))ds \\ -\int_0^{t_2}(t_1-s)^{\xi-1}v(s,x_n(s))ds\end{matrix}\right|$$

$$+\frac{1}{\Gamma(\xi)}\left|\begin{matrix}\int_0^{t_2}(t_1-s)^{\xi-1}v(s,x_n(s))ds \\ -\int_0^{t_1}(t_1-s)^{\xi-1}v(s,x_n(s))ds\end{matrix}\right|$$

$$\leq \frac{1}{\Gamma(\xi)}\left|\int_0^{t_2}(t_2-s)^{\xi-1}h(s)ds - \int_0^{t_2}(t_1-s)^{\xi-1}h(s)ds\right|$$

$$+\frac{1}{\Gamma(\xi)}\left|\int_0^{t_2}(t_1-s)^{\xi-1}h(s)ds - \int_0^{t_1}(t_1-s)^{\xi-1}h(s)ds\right|$$

$$\leq \frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi)}\left\{\begin{matrix}\left|\int_0^{t_2}\left[(t_2-s)^{\xi-1}-(t_1-s)^{\xi-1}\right]ds\right| \\ +\left|\int_{t_1}^{t_2}(t_1-s)^{\xi-1}ds\right|\end{matrix}\right\}$$

$$\leq \frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi)}\left\{\left|\left[\frac{(t_2-s)^{\xi}}{-\xi}\right]_0^{t_2} - \left[\frac{(t_1-s)^{\xi}}{-\xi}\right]_0^{t_2}\right| + \left|\left[\frac{(t_1-s)^{\xi}}{-\xi}\right]_{t_1}^{t_2}\right|\right\}$$

$$\leq \frac{\|h\|_{\mathcal{L}^1}}{\xi}\left\{\begin{matrix}\left|\begin{matrix}-\left[(t_2-t_2)^{\xi}-(t_2-0)^{\xi}\right]+ \\ \left[(t_1-t_2)^{\xi}-(t_1-0)^{\xi}\right]\end{matrix}\right| + \\ \left|-\left[(t_1-t_2)^{\xi}-(t_1-t_1)^{\xi}\right]\right|\end{matrix}\right\}$$

$$\leq \frac{\|h\|_{\mathcal{L}^1}}{\xi}\left\{\left|(t_2)^{\xi}+(t_1-t_2)^{\xi}-(t_1)^{\xi}\right| + \left|-(t_1-t_2)^{\xi}\right|\right\}$$

$$\to 0 \text{ as } t_1 \to t_2, \forall n \in \mathcal{N}$$

This shows that the Sequence $\{\mathbb{B}x_n\}$ uniformly convergence on S.

By using property of uniform convergence that is uniform convergence imply continuity.

Hence $\mathbb{B}$ is partially continuous on S.

Step IV: To show $\mathbb{B}$ is partially compact operator on S, for this to show that $\mathbb{B}$ is uniformly bounded and equicontinuous in S.

Let $\mathfrak{C}$ be an arbitrary chain in $\mathbb{E}$, then we show that $\mathbb{B}(\mathfrak{C})$ is uniformly bounded and equicontinuous set in S.

First we show that $\mathbb{B}(\mathfrak{C})$ is uniformly bounded set in S. Let $x \in \mathfrak{C}$ be arbitrary then

$$|\mathbb{B}x(t)| = \left|\frac{1}{\Gamma(\xi)}\int_0^{t}\frac{v(s,x(s))}{(t-s)^{1-\xi}}ds\right|$$

$$\leq \frac{1}{\Gamma(\xi)}\int_0^t \frac{|v(s,x(s))|}{(t-s)^{1-\xi}}ds$$

$$\leq \frac{1}{\Gamma(\xi)}\int_0^t \frac{h(s)}{(t-s)^{1-\xi}}ds$$

$$\leq \frac{\gamma(t)}{\Gamma(\xi)} = \mathcal{K}_1,\ \forall\, t \in \mathbb{R}_+$$

Taking supremum over t, we obtain $\|\mathbb{B}x\| \leq \mathcal{K}_1, \forall x \in \mathfrak{C}$.

This shows that $\mathbb{B}(\mathfrak{C})$ is uniformly bounded set in S.

Now we will show that $\mathbb{B}(\mathfrak{C})$is equicontinuous set in S.

Let $t_1, t_2 \in \mathbb{R}_+$with $t_1 < t_2$ then

$$|\mathbb{B}x(t_2) - \mathbb{B}x(t_1)| = \left|\frac{1}{\Gamma(\xi)}\int_0^{t_2}\frac{v(s,x(s))}{(t_2-s)^{1-\xi}}ds - \frac{1}{\Gamma(\xi)}\int_0^{t_1}\frac{v(s,x(s))}{(t_1-s)^{1-\xi}}ds\right|$$

$$\leq \frac{1}{\Gamma(\xi)}\left|\begin{matrix}\int_0^{t_2}(t_2-s)^{\xi-1}v(s,x(s))ds \\ -\int_0^{t_2}(t_1-s)^{\xi-1}v(s,x(s))ds\end{matrix}\right|$$

$$+\frac{1}{\Gamma(\xi)}\left|\begin{matrix}\int_0^{t_2}(t_1-s)^{\xi-1}v(s,x(s))ds \\ -\int_0^{t_1}(t_1-s)^{\xi-1}v(s,x(s))ds\end{matrix}\right|$$

$$\leq \frac{1}{\Gamma(\xi)}\left|\int_0^{t_2}(t_2-s)^{\xi-1}h(s)ds - \int_0^{t_2}(t_1-s)^{\xi-1}h(s)ds\right|$$

$$+\frac{1}{\Gamma(\xi)}\left|\int_0^{t_2}(t_1-s)^{\xi-1}h(s)ds - \int_0^{t_1}(t_1-s)^{\xi-1}h(s)ds\right|$$

$$\leq \frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi)}\left\{\begin{matrix}\left|\int_0^{t_2}\left[(t_2-s)^{\xi-1} - (t_1-s)^{\xi-1}\right]ds\right| \\ +\left|\int_{t_1}^{t_2}(t_1-s)^{\xi-1}ds\right|\end{matrix}\right\}$$

$$\leq \frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi)}\left\{\left|\left[\frac{(t_2-s)^{\xi}}{-\xi}\right]_0^{t_2} - \left[\frac{(t_1-s)^{\xi}}{-\xi}\right]_0^{t_2}\right| + \left|\left[\frac{(t_1-s)^{\xi}}{-\xi}\right]_{t_1}^{t_2}\right|\right\}$$

$$\leq \frac{\|h\|_{\mathcal{L}^1}}{\xi}\left\{\begin{matrix}\left|\begin{matrix}-\left[(t_2-t_2)^{\xi} - (t_2-0)^{\xi}\right] + \\ \left[(t_1-t_2)^{\xi} - (t_1-0)^{\xi}\right]\end{matrix}\right| + \\ \left|-\left[(t_1-t_2)^{\xi} - (t_1-t_1)^{\xi}\right]\right|\end{matrix}\right\}$$

$$\leq \frac{\|h\|_{\mathcal{L}^1}}{\xi}\left\{\left|(t_2)^{\xi} + (t_1-t_2)^{\xi} - (t_1)^{\xi}\right| + \left|-(t_1-t_2)^{\xi}\right|\right\}$$

$$\to 0 \text{ as } t_1 \to t_2, \forall n \in \mathcal{N}$$

This shows that $\mathbb{B}(\mathfrak{C})$ is equicontinuous set in S and so $\mathbb{B}(\mathfrak{C})$ is relatively compact by Arzela Ascoli theorem.

Hence $\mathbb{B}(\mathfrak{C})$ is compact subset of S and consequently $\mathbb{B}$ is partially compact operator on S.

Step V: To show $x = \mathbb{A}x\mathbb{B}x + \mathbb{C}x \in S, \forall x \in S$.

Let $x \in S$ be an arbitrary element such that $x = \mathbb{A}x\mathbb{B}x + \mathbb{C}x$,

Then we have

$$|x(t)| = |\mathbb{A}x(t)\mathbb{B}x(t) + \mathbb{C}x(t)|$$

$$\leq |\mathbb{A}x(t)||\mathbb{B}x(t)| + |\mathbb{C}x(t)|$$

$$\leq \left|f\big(t, x(t)\big)\right| \left|\frac{1}{\Gamma(\xi)} \int_0^t \frac{v\big(s, x(s)\big)}{(t-s)^{1-\xi}} ds\right| + \left|q\big(t, x(t)\big)p(t)\right|$$

$$\leq \left|f\big(t, x(t)\big)\right| \frac{1}{\Gamma(\xi)} \int_0^t \frac{\left|v\big(s, x(s)\big)\right|}{(t-s)^{1-\xi}} ds + \left|q\big(t, x(t)\big)p(t)\right|$$

$$\leq \frac{\mathbb{F}}{\Gamma(\xi)} \int_0^t \frac{h(s)}{(t-s)^{1-\xi}} ds + \left|q\big(t, x(t)\big)p(t)\right|$$

$$\leq \mathbb{F}\frac{\gamma(t)}{\Gamma(\xi)} + \left|q\big(t, x(t)\big)p(t)\right|$$

$$\leq \mathbb{F}\mathcal{K}_1 + \left|q\big(t, x(t)\big)\right||p(t)|$$

$$\leq \mathbb{F}\mathcal{K}_1 + \mathbb{Q}\mathbb{P}$$

$$\leq \mathbb{F}\mathcal{K}_1 + \mathcal{K}_2 \leq \mathfrak{z} > 0, \forall t \in \mathbb{R}_+$$

Taking supremum over t, we obtain $\|x\| \leq \mathfrak{z}$,

Therefore $x \in S$.

Step VI: Also we have

$$M = \|\mathbb{B}(\mathfrak{C})\| = sup\{\|\mathbb{B}x\| : x \in \mathfrak{C}\}$$

$$= sup\left\{sup_{t\geq 0}\left\{\frac{1}{\Gamma(\xi)} \int_0^t \frac{\left|v\big(s, x(s)\big)\right|}{(t-s)^{1-\xi}} ds\right\} : x \in \mathfrak{C}\right\}$$

$$= sup\left\{sup_{t\geq 0}\left\{\frac{1}{\Gamma(\xi)} \int_0^t \frac{h(s)}{(t-s)^{1-\xi}} ds\right\} : x \in \mathfrak{C}\right\}$$

$$\leq sup_{t\geq 0}\frac{\gamma(t)}{\Gamma(\xi)} = \mathcal{K}_1$$

Then by theorem (3.1),

$$M\varphi_{\mathbb{A}}(r) + \varphi_{\mathbb{C}}(r) = \mathcal{K}_1\|\ell_{\mathbb{A}}\| + \mathbb{P}\|\ell_{\mathbb{C}}\| < r, r > 0, \text{where } M = \|\mathbb{B}(\mathfrak{C})\|$$

Step VII: Here the function $p: \mathbb{R}_+ \to \mathbb{R}_+$, $q, f, v: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, are continuous, nonnegative, nondecreasing for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}_+$.

The operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are monotonic increasing that is nondecreasing.

Now by hypothesis ($\boldsymbol{\mathcal{H}_6}$), that is $u(t) \leq q\big(t, u(t)\big)p(t) + \frac{f(t,u(t))}{\Gamma(\xi)} \int_0^t \frac{v(s,u(s))}{(t-s)^{1-\xi}} ds$

Therefore u satisfies the operator inequality $u \leq \mathbb{A}u\mathbb{B}u + \mathbb{C}u$

There exist an element $x_0 = u \in \mathfrak{C} \subset S$ such that $x_0 \preccurlyeq \mathbb{A}x_0\mathbb{B}x_0 + \mathbb{C}x_0$

Thus the operators $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ satisfies all the conditions of theorem (3.1) on S. Hence the operator equation $x(t) = \mathbb{A}x(t)\mathbb{B}x(t) + \mathbb{C}x(t)$ has a solution in S.Clearly, x^* is a solution of the FQIE (1.2) and the sequence $\{x_n\}$ of successive approximations defined by

$$x_{n+1}(t) = q(t, x_n(t))p(t) + \frac{f(t,x_n(t))}{\Gamma(\xi)}\int_0^t \frac{v(s,x_n(s))}{(t-s)^{1-\xi}}ds, t \in \mathbb{R}_+,$$

converges monotonically to x^*.

Therefore the FQIE (1.2) has a solution x_0 defined on $\mathbb{R}_+$.

Step VII: Now to show the solution is locally attractive on $\mathbb{R}_+$. Then we have

$$|x(t) - y(t)| = \left| \begin{array}{l} \left\{q(t, x(t))p(t) + \frac{f(t,x(t))}{\Gamma(\xi)}\int_0^t \frac{v(s,x(s))}{(t-s)^{1-\xi}}ds\right\} - \\ \left\{q(t, y(t))p(t) + \frac{f(t,y(t))}{\Gamma(\xi)}\int_0^t \frac{v(s,y(s))}{(t-s)^{1-\xi}}ds\right\} \end{array} \right|$$

$$\leq |q(t, x(t))p(t)| + \frac{|f(t, x(t))|}{\Gamma(\xi)}\int_0^t \frac{|v(s, x(s))|}{(t-s)^{1-\xi}}ds$$

$$+|q(t, y(t))p(t)| + \frac{|f(t, y(t))|}{\Gamma(\xi)}\int_0^t \frac{|v(s, y(s))|}{(t-s)^{1-\xi}}ds$$

$$\leq 2\mathbb{P}|q(t, y(t))| + 2\mathbb{F}\frac{\gamma(t)}{\Gamma(\xi)}, \forall t \in \mathbb{R}_+$$

Since $lim_{t\to\infty}\gamma(t) = 0$ and $lim_{t\to\infty}q(t, x(t)) = 0$

for $\epsilon > 0$, there is real number $\mathbb{T}' > 0, \mathbb{T}'' > 0$ such that $\gamma(t) \leq \frac{\Gamma(\xi)\epsilon}{4F}$ for all $t \geq \mathbb{T}^*$ and $|q(t, x(t))| < \frac{\epsilon}{4\mathbb{P}}$. If we choose $\mathbb{T}^* = max\{\mathbb{T}', \mathbb{T}''\}$

Then from above inequality it follows that $|x(t) - y(t)| < \epsilon$ for all $t \geq \mathbb{T}^*$.

Hence FQIE (1.2) has a locally attractive solution on $\mathbb{R}_+$.

This completes the proof.

V. Example:

Example5.1.: Consider the following quadratic integral equation of fractional order

$$x(t) = t^2x(t).c(1 - e^{-at}) + \frac{tx(t)}{2}\frac{1}{\Gamma(\xi)}\int_0^t \frac{\frac{t^2x^2+3}{(t^2+1)}}{(t-s)^{1-\xi}}ds\,, \quad a > 0 \tag{5.1}$$

Where $t \in \mathbb{R}_+$and $\xi \in (0,1)$ be a fixed number.

Notice that this equation is special case of equation (1.2)

Solution Here, $q(t, x) = t^2x(t), p(t) = c(1 - e^{-at}), a > 0, f(t, x) = \frac{tx}{2}$,

$$v(t, x) = \frac{t^2x^2+3}{(t^2+1)}$$

It is easy to check that for equation (5.1) there are satisfied all the conditions of theorem (4.1).

To prove this assertion observe that the function $q(t, x)$, $p(t)$,$f(t, x)$, $v(s, x)$ positive, nondecreasing and continuous on $t \in \mathbb{R}_+$,

a) To prove the hypothesis $(\mathcal{H}_1)$ is satisfied:

Here $p(t) = c(1 - e^{-at}) \leq c\,, \forall t \in \mathbb{R}_+$

Hence assumption $(\mathcal{H}_1)$ hold

b) To prove the hypothesis ($\mathcal{H}_2$) is satisfied

Let $\left|f\big(t,x(t)\big)-f\big(t,y(t)\big)\right| = \left|\frac{tx(t)}{2}-\frac{ty(t)}{2}\right| \leq \frac{t}{2}|x(t)-y(t)|$

$$\left|f\big(t,x(t)\big)-f\big(t,y(t)\big)\right| \leq \ell_{\mathbb{A}}(t)|x(t)-y(t)| \left(\because \ell_{\mathbb{A}}(t)=\frac{t}{2}\right)$$

Henceassumption($\mathcal{H}_2$) hold.

c) Now

$$\left|q\big(t,x(t)\big)p(t)-q\big(t,y(t)\big)p(t)\right| = \left|t^2x(t)c(1-e^{-at})-t^2y(t)c(1-e^{-at})\right|$$

$$\leq \left|t^2c(1-e^{-at})\right||x(t)-y(t)|$$

$$\leq \ell_{\mathbb{C}}(t)|x(t)-y(t)|$$

$$\left(\because \ell_{\mathbb{C}}(t)=t^2c(1-e^{-at})\right)$$

Hence assumption ($\mathcal{H}_3$) hold.

d) To show hypothesis ($\mathcal{H}_4$) is satisfied:

That is $v(t,x)=\frac{t^2x^2+3}{(t^2+1)} \leq \frac{1}{(t^2+1)} = h(t)$

Now based on theorem (4.1) we conclude that equation (5.1) has positive, nondecreasing solution $x=x(t)$ for $t,x \in \mathbb{R}_+$.

Conclusion

In this paper we have studied the existence of solution for the quadratic integral equation of fractional order. The result has been obtained by using hybrid fixed point theorem for three operators in partially ordered normed linear space due to Dhage. The main result is well illustrated with the help of example.

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