Integral solutions of Ternary Cubic Diophantine equation $x^3 - xy + y^3 = 4z^3$

R. Anbuselvi
Associate Professor, Department of Mathematics, ADM College for women, Nagapattinam, Tamilnadu, India

S. Jamuna Rani
Asst Professor, Department of Computer Applications, Bharathiyar college of Engineering and Technology, Karaikal, Puducherry, India

Abstract
The ternary cubic Diophantine equation given by $x^3 - xy + y^3 = 4z^3$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords
Ternary cubic, integral solutions, polygonal numbers.

Introduction
Ternary quadratic equations are rich in variety [1-3]. For an extensive review of sizable literature and various problems, one may refer [4-18]. In this communication, we consider yet another interesting ternary cubic equation $x^3 - xy + y^3 = 4z^3$ and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions and special polygonal numbers are presented.

Notations Used
- $t_{m,n}$ - Polygonal number of rank ‘n’ with size ‘m’
- $CP_{m,n}$ - Centered Pyramidal number of rank ‘n’ with size ‘m’
- $P_n$ - Pronic number of rank ‘n’
- $P^2_n$ - Pyramidal number of rank ‘n’ with size ‘m’
- $F_{m,n}$ - Figurative number of rank ‘n’ with size ‘m’
- $G_n$ - Gnomonic number of rank ‘n’

Method of Analysis
The Cubic Diophantine equation with three unknowns to be solved for its non-zero distinct integral solutions is

$$x^3 - xy + y^3 = 4z^3 \quad (1)$$

We illustrate methods of obtaining non-zero distinct integral solutions to (1)

Pattern I
On substituting the linear transformations

$$x = u + v \quad y = u - v \quad (2)$$

in (1), leads to

$$u^2 + 3v^2 = 4z^3 \quad (3)$$

Assume

$$z = z(a, b) = a^2 + 3b^2 ; \quad a, b > 0 \quad (4)$$

Write 4 as

$$4 = \frac{(2.2^n + i.2^n.2\sqrt{3})(2.2^n - i.2^n.2\sqrt{3})}{2^{n+1}} \quad (5)$$

Equation (3) can be written as

$$(u + i\sqrt{3}v)(u - i\sqrt{3}v) = \frac{(2.2^n + i.2^n.2\sqrt{3})(2.2^n - i.2^n.2\sqrt{3})(a + i\sqrt{3}b)(a - i\sqrt{3}b)}{2^{n+1}}$$

Which is equivalent to the system of equations

$$u + i\sqrt{3}v = \frac{(2.2^n + i.2^n.2\sqrt{3})(a + i\sqrt{3}b)}{2^{n+1}} \quad (6)$$

Equating real and imaginary parts in (6) we get

$$u = \frac{2^n}{2^{n+1}} (2a^3 + 18b^3 - 18a^2b - 18ab^2)$$

$$v = \frac{2^n}{2^{n+1}} (2a^3 - 6b^3 + 6a^2b - 18ab^2)$$

Substituting the values of u and v into the values of x and y are given by

$$x = x(a, b) = 2a^3 + 6b^3 - 6a^2b - 18ab^2$$

$$y = y(a, b) = 12b^3 - 12a^2b$$

$$z = z(a, b) = a^2 + 3b^2 \quad (7)$$

Properties
1. $2x(a, 0) - y(a, 0) - 4CP_a^6 \equiv 0$
2. $2x(a, 1) - y(a, 1) - 2SO_4 \equiv 0 (mod 34)$
3. \(2x(1,b) - y(1,b) + 36 t_{4,b} - 4 \equiv 0\)
4. \(x(1,b) + y(1,b) - 185O_4 + 18CP \alpha^6 + 18t_{4,b} - 2 \equiv 0\)
5. \(x(\alpha, 1) - y(\alpha, 1) - 5O_\alpha - 6Pr_\alpha \equiv 6(\mod 23)\)
6. \(y(1,b) + z(1,b) - 6P_\alpha - 50_\alpha - 7CP \alpha^6 + Gno \equiv 0(\mod 9)\)

7. Each of the following expression represents a cube integer
   a. \(y(1,4) + x(0,1) + z(0,1)\)
   b. \(z(0,1) - x(4,4) - x(4,2)\)
   c. \(x(1,0) + x(2,3)\)
   d. \(z(4,4)\)
   e. \(y(1,2) - x(3,3)\)

8. Each of the following expression represents a perfect square numbers
   a. \(x(0,1) - x(3,3) + z(0,1)\)
   b. \(x(1,0) - x(1,2)\)
   c. \(z(0,1) - x(3,2)\)
   d. \(y(1,4) + z(4,4)\)

9. Each of the following expression represents a nasty number
   a. \(x(0,1) - x(3,1)\)
   b. \(z(3,3) - y(1,3)\)
   c. \(z(3,2) - z(3,1)\)
   d. \(x(3,2) - y(4,3)\)
   e. \(\frac{1}{6} y(1,4)\)

10. Each of the following expression represents a perfect number
    a. \(z(1,3)\)
    b. \(y(1,3) + z(1,1)\)
    c. \(508 z(2,2)\)
    d. \(y(1,4) + x(2,3) - \frac{1}{2} z(3,1)\)
    e. \(\frac{1}{2} x(4,4) - z(2,2)\)
    f. \(\frac{1}{2} x(4,4) + x(1,1)\)

**Pattern II**

On substituting the linear transformation

\[x = 2u + v; \quad y = 2u - v; \quad z = u\]  \(8\)

In (1) leads to

\[v^2 = \frac{4u^2(u - 1)}{3}\]  \(9\)

Take \(\alpha^2 = \frac{u-1}{3}\)

which implies the values

\[u = 3\alpha^2 + 1\]  and \[v = 2(3\alpha^2 + 1)\alpha\]  \(10\)

Substituting the values of u and v in (8) the values of x and y are given by

\[x = x(\alpha) = 6\alpha^3 + 6\alpha^2 + 2\alpha + 12\]
\[y = y(\alpha) = 6\alpha^2 - 6\alpha^3 - 2\alpha + 2\]
\[z = z(\alpha) = 3\alpha^2 + 1\]  \(11\)

**Properties**

1. \(x(\alpha) + y(\alpha) - 12 t_{4,\alpha} - 14 \equiv 0\)
2. \(x(\alpha) - y(\alpha) - 12 CP \alpha^6 - 2Gno - 12 \equiv 0\)
3. \(x(\alpha) - y(\alpha) - 6SO_\alpha - Gno - 15 \equiv 0\)
4. \(y(\alpha) - 2z(\alpha) + 6CP \alpha^6 - Gno + 1 \equiv 0\)
5. \(y(\alpha) - 2z(\alpha) + 3SO_\alpha + 5\alpha \equiv 0\)
6. \(y(\alpha) + 2z(\alpha) + CP \alpha^6 - 12 t_{4,\alpha} + Gno - 3 \equiv 0\)
7. \(y(\alpha) - 2z(\alpha) - 12P \alpha^6 - 10 \equiv 0\)
8. \(x(\alpha) - 2z(\alpha) - 6 CP \alpha^6 - 6 t_{4,\alpha} - 10 \equiv 0\)
9. \(x(\alpha) - 2z(\alpha) - 6 CP \alpha^6 - 6 Pr_\alpha + 3Gno - 7 \equiv 0\)
10. \(x(\alpha) + 2z(\alpha) - 12P \alpha^6 - 2Pr_\alpha - 4 t_{4,\alpha} - 14 \equiv 0\)
11. \(x(\alpha) + 2z(\alpha) - 6CP \alpha^6 - 2Pr_\alpha - 10 t_{4,\alpha} - 14 \equiv 0\)
12. \(x(\alpha) + y(\alpha) + z(\alpha) - 15 t_{4,\alpha} - 15 \equiv 0\)
13. \(x(\alpha) - y(\alpha) + z(\alpha) - 24 P \alpha^6 + 9Pr_\alpha \equiv 11(\mod 13)\)
14. \(x(\alpha) - y(\alpha) + z(\alpha) - 12 CP \alpha^6 - 3 t_{4,\alpha} \equiv 1(\mod 4)\)
15. \(x(\alpha) - y(\alpha) + z(\alpha) - 12 CP \alpha^6 - 3Pr_\alpha - \alpha - 11 \equiv 0\)

16. Each of the following expression represents a perfect number
    a. \(x(1) + y(0)\)
    b. \(y(-4) + z(1) + y(0)\)
    c. \(x(-3) - x(-5)\)
    d. \(y(3) - y(5)\)
    e. \(x(4) - z(1)\)
    f. \(y(-3) - x(-4) - x(0)\)
    g. \(16[y(-4) + y(-1) + y(0)]\)

17. Each of the following expression represents a nasty number
    a. \(z(-3) + y(0)\)
    b. \(y(-2) + z(-3)\)
    c. \(x(1) + z(1)\)
    d. \(z(-2) - x(-2) + z(0)\)
    e. \(-y(3) + 2z(1)\)
    f. \(\frac{1}{2} [y(-3) + z(-5)]\)
    g. \(y(-2) - y(3)\)
    h. \(z(-3) - z(1)\)
    i. \(x(2) - z(-3)\)

18. Each of the following expression represents a cube number
    a. \(x(4) + x(0)\)
    b. \(x(4) + y(-3) + z(-1) + z(0)\)
c. \(x(4) + y(-4) + x(-1)\)
d. \(y(-3) - x(-4) + z(1)\)
e. \(x(5) - y(5) + y(-3) - y(4) + 2z(1)\)

19. Each of the following expression represents a perfect square
   a. \(y(-3) + z(0)\)
   b. \(x(4) + z(-3) + z(0)\)
   c. \(y(-5) - x(0)\)
   d. \(y(-5) - z(4)\)
   e. \(y(-5) - y(3)\)
   f. \(y(-2) - y(5)\)
   g. \(x(4) + y(-4) - x(-5) + x(0)\)

Pattern III
Consider the another linear transformation
\[x = u + 2v; \quad y = u - 2v; \quad z = v\] (12)
On substituting these values in (1) leads to
\[u^2 = 4v^2 (v - 3)\] (13)
Take \(\beta^2 = v - 3\) which implies the values
\[u = 2\beta^2 + 6\beta\]
\[v = \beta^2 + 3\] (14)
Substituting the values of \(u\) and \(v\) in (12), the values of \(x\) and \(y\) are obtained. we have
\[x = x(\beta) = 2\beta^3 + 2\beta^2 + 6\beta + 6\]
\[y = y(\beta) = 2\beta^3 - 2\beta^2 + 6\beta - 6 + z = z(\beta) = \beta^2 + 3\] (15)

Properties
1. \(x(\beta) + y(\beta) - 4CP^0_{\beta} \equiv 0 (\text{mod} 12)\)
2. \(x(\beta) + y(\beta) - 2SO_{\beta} \equiv 0 (\text{mod} 14)\)
3. \(x(\beta) - y(\beta) - 4PR_{\beta} - 8 \equiv 0\)
4. \(x(\beta) - y(\beta) - 4t_{4,\beta} + 12 \equiv 0\)
5. \(x(\beta) + 2y(\beta) - 2CP^0_{\beta} - 4t_{4,\beta} + 12 (\text{mod} 6)\)
6. \(x(\beta) + 2y(\beta) - SO_{\beta} - 4PR_{\beta} \equiv 12 (\text{mod} 3)\)
7. \(x(\beta) + 2y(\beta) - 4P^1_{\beta} - 2PR_{\beta} - 2Gno - 14 \equiv 0\)
8. \(x(\beta) - 2z(\beta) - SO_{\beta} \equiv 0 (\text{mod} 7)\)
9. \(x(\beta) - 2x(\beta) - 2CP^0_{\beta} \equiv 0 (\text{mod} 6)\)
10. \(y(\beta) - 2x(\beta) - SO_{\beta} + 4PR_{\beta} \equiv -12 (\text{mod} 11)\)
11. \(y(\beta) - 2x(\beta) - 2CP^1_{\beta} + 4t_{4,\beta} - 3Gno + 9 \equiv 0\)
12. \(y(\beta) - 2x(\beta) - 4P^1_{\beta} + 6PR_{\beta} - 6Gno + 6 \equiv 0\)
13. \(x(\beta) + y(\beta) + 2z(\beta) + 2SO_{\beta} - 2PR_{\beta} \equiv 6 (\text{mod} 12)\)
14. \(x(\beta) + y(\beta) - 2z(\beta) - 2SO_{\beta} + 2PR_{\beta} - 8Gno - 2 \equiv 0\)
15. \(x(\beta) + y(\beta) - 2z(\beta) - 4P^1_{\beta} + 6PR_{\beta} - 9Gno - 3 \equiv 0\)
16. \(x(\beta) + y(\beta) - 2z(\beta) - 4CP^1_{\beta} + 2t_{4,\beta} \equiv -6 (\text{mod} 12)\)
17. \(x(\beta) + y(\beta) + 2z(\beta) - 6t_{4,\beta} - 18 \equiv 0\)
18. Each of the following expression represents a perfect number
   a. \(x(5) + y(5) - z(1)\)
   b. \(x(4) - y(-4) + y(4) + \frac{1}{2}z(1)\)
   c. \(32y(5) + 96[x(-2) + z(0)]\)
19. Each of the following expression represents a nasty number
   a. \(y(3) + x(0)\)
   b. \(y(4) + x(0)\)
   c. \(\frac{1}{2}y(3)\)
   d. \(x(4) - x(0) - z(1)\)
   e. \(2y(3) + x(2) + z(-3)\)
20. Each of the following expression represents a cube number
   a. \(z(3) - z(-1)\)
   b. \(y(5) - 2z(1)\)
   c. \(x(5) + z(-2)\)
   d. \(x(5) + y(5) - y(3)\)
   e. \(2[x(5) + y(5) + x(4) + y(4)]\)
21. Each of the following expression represents a perfect square
   a. \(x(3) + z(1)\)
   b. \(x(2) + z(2)\)
   c. \(x(1) + y(2)\)
   d. \(x(2) + y(0)\)
   e. \(x(4) + x(0)\)
   f. \(x(5) + y(5) + x(1)\)
   g. \(x(5) + y(5) + x(4) + y(4) + 3z(-3)\)
   h. \(y(4) + x(3) + x(2) + z(-1)\)
   i. \(2x(5) + z(1)\)
   j. \(2y(5) + 3z(3)\)

Conclusion
In this paper, we have presented three different patterns of non-zero distinct integer solutions of ternary cubic Diophantine equation \(x^2 - xy + y^2 = 4z^3\) and relations between solutions and special numbers are also obtained. To conclude, one may search for other patterns of solutions and their corresponding properties.

REFERENCES
Journal Articles


Reference Books

