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Integral solutions of Ternary Cubic Diophantine equation x^2-xy+y^2=$4z^3$

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ABSTRACT The ternary cubic Diophantine equation given by $x^2 - xy + y^2 = 4z^3$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

KEYWORDS Ternary cubic, integral solutions, polygonal numbers.

Introduction

Ternary quadratic equations are rich in variety [1-3]. For an extensive review of sizable literature and various problems, one may refer [4-18]. In this communication, we consider yet another interesting ternary cubic equation $x^2 - xy + y^2 = 4z^3$ and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions and special polygonal numbers are presented.

Notations Used

- $t_{m,n}$- Polygonal number of rank 'n' with size 'm'
- $CP_{m,n}$- Centered Pyramidal number of rank 'n' with size 'm'
- Pr_n - Pronic number of rank 'n'
- P_n^m- Pyramidal number of rank 'n' with size 'm'
- $F_{m,n}$ – Figurative number of rank 'n' with size 'm'
- G_n – Gnomic number of rank 'n'

Methodof Analysis

The Cubic Diophantine equation with three unknowns to be solved for its non zero distinct integral solutions is

$$x^2 - xy + y^2 = 4z^3 \qquad (1)$$

We illustrate methods of obtaining non Zero distinct integer solutions to (1)

Pattern I

On substituting the linear transformations

$$x = u + v \qquad y = u - v \qquad (2)$$

in (1), leads to

$$u^2 + 3v^2 = 4z^3 \qquad (3)$$

Assume

$$z = z(a,b) = a^2 + 3b^2 \; ; \qquad a,b > 0 \qquad (4)$$

Write 4 as

$$4 = \frac{(2.2^n + i.2^n.2\sqrt{3})\,(2.2^n - i.2^n.2\sqrt{3})}{2^{n+1}} \qquad (5)$$

Equation (3) can be written as

$$(u + i\sqrt{3}\,v)(u - i\sqrt{3}\,v) = \frac{(2.2^n + i.2^n.2\sqrt{3})(2.2^n - i.2^n.2\sqrt{3})[(a + i\sqrt{3}\,b)(a - i\sqrt{b})\,]^3}{2^{n+1}}$$

Which is equivalent to the system of equations

$$(u + i\sqrt{3}\,v) = \frac{(2.2^n + i.2^n.2\sqrt{3})[(a + i\sqrt{3}\,b)\,]^3}{2^{n+1}} \qquad (6)$$

Equating real and imaginary parts in (6) we get

$$u = \frac{2^n}{2^{n+1}}\,(2a^3 + 18b^3 - 18a^2b - 18ab^2)$$

$$v = \frac{2^n}{2^{n+1}}\,(2a^3 - 6b^3 + 6a^2b - 18ab^2)$$

Substituting the values of u and v into the values of x and y are given by

$$\left.\begin{aligned} x = x(a,b) &= 2a^3 + 6b^3 - 6a^2b - 18ab^2 \\ y = y(a,b) &= 12b^3 - 12a^2b \\ z = z(a,b) &= a^2 + 3b^2 \end{aligned}\right\} \qquad (7)$$

Properties

1. $2x(a,0) - y(a,0) - 4CP_a^6 \equiv 0$
2. $2x(a,1) - y(a,1) - 2SO_4 \equiv 0(mod\ 34)$

3. $2x(1,b) - y(1,b) + 36\, t_{4,b} - 4 \equiv 0$
4. $x(1,b) + y(1,b) - 18SO_4 + 18CP_b^6 + 18t_{4,b} - 2 \equiv 0$
5. $x(a,1) - y(a,1) - SO_a - 6Pr_a \equiv 6(mod\ 23)$
6. $y(1,b) + z(1,b) - 6\, P_b^5 - SO_b - 7CP_a^6 + Gno \equiv 0(mod\ 9)$
7. Each of the following expression represents a cube integer
   a. $y(1,4) + x(0,1) + z(0,1)$
   b. $z(0,1) - x(4,4) - x(4,2)$
   c. $x(1,0) + x(2,3)$
   d. $z(4,4)$
   e. $y(1,2) - x(3,3)$
8. Each of the following expression represents a perfect square numbers
   a. $x(0,1) - x(3,3) + z(0,1)$
   b. $x(1,0) - x(1,2)$
   c. $z(0,1) - x(3,2)$
   d. $y(1,4) + z(4,4)$
9. Each of the following expression represents a nasty number
   a. $x(0,1) - x(3,1)$
   b. $z(3,3) - y(1,3)$
   c. $y(3,2) - z(3,1)$
   d. $x(3,2) - y(4,3)$
   e. $\frac{1}{6}\, y(1,4)$
10. Each of the following expression represents a perfect number
   a. $z(1,3)$
   b. $y(1,3) + z(1,1)$
   c. $508\, z(2,2)$
   d. $y(1,4) + x(2,3) - \frac{1}{2}\, z(3,1)$
   e. $\frac{1}{2}\, x(4,4) - z(2,2)$
   f. $\frac{1}{2}\, x(4,4) + x(1,1)$

Pattern II

On substituting the linear transformation

$$x = 2u + v\,; \quad y = 2u - v\,; \quad z = u \tag{8}$$

in (1) leads to

$$v^2 = \frac{4u^2(u-1)}{3} \tag{9}$$

Take $\alpha^2 = \frac{u-1}{3}$

which implies the values

$$u = 3\alpha^2 + 1 \text{ and } v = 2(3\alpha^2 + 1)\alpha \tag{10}$$

Substituting the values of u and v in (8) the values of x and y are given by

$$\left.\begin{aligned} x = x(\alpha) &= 6\alpha^3 + 6\alpha^2 + 2\alpha + 12 \\ y = y(\alpha) &= 6\alpha^2 - 6\alpha^3 - 2\alpha + 2 \\ z = z(\alpha) &= 3\,\alpha^2 + 1 \end{aligned}\right\} \tag{11}$$

Properties

1. $x(\alpha) + y(\alpha) - 12\, t_{4,\alpha} - 14 \equiv 0$
2. $x(\alpha) - y(\alpha) - 12\, CP_\alpha^6 - 2\, Gno - 12 \equiv 0$
3. $x(\alpha) - y(\alpha) - 6SO_\alpha - Gno - 15 \equiv 0$
4. $y(\alpha) - 2z(\alpha) + 6CP_\alpha^6 - Gno + 1 \equiv 0$
5. $y(\alpha) - 2z(\alpha) + 3SO_\alpha + 5\alpha \equiv 0$
6. $y(\alpha) + 2z(\alpha) + CP_\alpha^6 - 12\, t_{4,\alpha} + Gno - 3 \equiv 0$
7. $y(\alpha) - 2z(\alpha) - 12P_\alpha^5 - 10 \equiv 0$
8. $x(\alpha) - 2z(\alpha) - 6\, CP_\alpha^6 - 6\, t_{4,\alpha} - 10 \equiv 0$
9. $x(\alpha) - 2z(\alpha) - 6\, CP_\alpha^6 - 6Pr_\alpha + 3Gno - 7 \equiv 0$
10. $x(\alpha) + 2z(\alpha) - 12Pa_\alpha^5 - 2Pr_\alpha - 4\, t_{4,\alpha} - 14 \equiv 0$
11. $x(\alpha) + 2z(\alpha) - 6CP_\alpha^6 - 2Pr_\alpha - 10\, t_{4,\alpha} - 14 \equiv 0$
12. $x(\alpha) + y(\alpha) + z(\alpha) - 15\, t_{4,\alpha} - 15 \equiv 0$
13. $x(\alpha) - y(\alpha) + z(\alpha) - 24\, Pa_\alpha^5 + 9Pr_\alpha \equiv 11(mod\ 13)$
14. $x(\alpha) - y(\alpha) + z(\alpha) - 12\, CP_\alpha^6 - 3\, t_{4,\alpha} \equiv 1(mod\ 4)$
15. $x(\alpha) - y(\alpha) + z(\alpha) - 12\, CP_\alpha^6 - 3Pr_\alpha - \alpha - 11 \equiv 0$
16. Each of the following expression represents a perfect number
   a. $x(1) + y(0)$
   b. $y(-4) + z(1) + y(0)$
   c. $x(-3) - x(-5)$
   d. $y(3) - y(5)$
   e. x(4)-z(1)
   f. $y(-3) - x(-4) - x(0)$
   g. $16[y(-4) + y(-1) + y(0)]$
17. Each of the following expression represents a nasty number
   a. $z(-3) + y(0)$
   b. $y(-2) + z(-3)$
   c. $x(1) + z(1)$
   d. $z(-2) - x(-2) + z(0)$
   e. $-y(3) + 2z(1)$
   f. $\frac{1}{2}\, [y(-3) + z(-5)]$
   g. $y(-2) - y(3)$
   h. $z(-3) - z(1)$
   i. $x(2) - z(-3)$
18. Each of the following expression represents a cube number
   a. $x(4) + x(0)$
   b. $x(4) + y(-3) + z(-1) + z(0)$

c. $x(4)+y(-4)+x(-1)$

d. $y(-3)-x(-4)+z(1)$

e. x(5)-y(5)+y(-5)-y(4)+2z(1)

19. Each of the following expression represents a perfect square

a. $y(-3)+z(0)$

b. $x(4)+z(-3)+z(0)$

c. $y(-5)-x(0)$

d. $y(-5)-z(4)$

e. $y(-5)-y(3)$

f. $y(-2)-y(5)$

g. $x(4)+y(-4)-x(-5)+x(0)]$

Pattern III

Consider the another linear transformation

$$x=u+2v; \quad y=u-2v\,; z=v \qquad (12)$$

On substituting these values in (1) leads to

$$u^2=4v^2\,(v-3) \qquad (13)$$

Take $\beta^2=v-3$

which implies the values

$$u=2\beta^2+6\beta$$
$$v=\beta^2+3 \qquad (14)$$

Substituting the values of u and v in (12), the values of x and y are obtained. we have

$$\left.\begin{aligned} x= x(\beta)&=2\beta^3+2\beta^2+6\beta+6\\ y=y(\beta)&=2\beta^3-2\beta^2+6\beta-6\\ z=z(\beta)&=\beta^2+3\end{aligned}\right\} \qquad (15)$$

Properties

1. $x(\beta)+y(\beta)-4CP_\beta^6\equiv 0(mod\ 12)$
2. $x(\beta)+y(\beta)-2\,So_\beta\ \equiv 0\ (mod\ 14)$
3. $x(\beta)-y(\beta)-4Pr_\beta-8\equiv 0$
4. $x(\beta)-y(\beta)-4\,t_{4,\beta}\ -12\equiv 0$
5. $x(\beta)+2z(\beta)-\ 2\,CP_\beta^6-4\,t_{\,4,\beta}\ \equiv 12(mod\ 6)$
6. $x(\beta)+2z(\beta)-SO_\beta-4\,Pr_\beta\equiv 12(\text{mod } 3)$
7. $x(\beta)+2z(\beta)-4\,P_\beta^5-2\,Pr_\beta-2Gno-14\equiv 0$
8. $x(\beta)-2z(\beta)-So_\beta\equiv 0\ (mod\ 7)$
9. $x(\beta)-2z(\beta)-2CP_\beta^6\equiv 0\ (mod\ 6)$
10. $y(\beta)-2z(\beta)-SO_\beta+4Pr_\beta\equiv -12\ (mod\ 11)$
11. $y(\beta)-2z(\beta)-2CP_\beta^6+4\,t_{\,4\,,\beta}-3Gno+9\equiv 0$
12. $y(\beta)-2z(\beta)-4\,P_\beta^5+6Pr_\beta-6Gno+6\equiv 0$
13. $x(\beta)+y(\beta)+2z(\beta)-2\,SO_\beta-2\,Pr_{\,\beta}\equiv 6\ (mod\ 12)$
14. $x(\beta)+y(\beta)-2z(\beta)-2SO_\beta+2Pr_\beta-8Gno-2\equiv 0$
15. $x(\beta)+y(\beta)-2z(\beta)-4P_\alpha^5+6Pr_\beta-9Gno-3\equiv 0$
16. $x(\beta)+y(\beta)-2z(\beta)-4CP_\beta^6+2t_{\,4,\beta}\equiv -6\ (mod\ 12)$
17. $x(\beta)+y(\beta)+2z(\beta)-6\,t_{\,4\ \beta}-18\equiv 0$
18. Each of the following expression represents a perfect number

a. $x(5)+y(5)-z(1)$

b. $x(4)-y(-4)+y(4)+\frac{1}{2}z(1)$

c. $32y(5)+96[z(-2)+z(0)]$

19. Each of the following expression represents a nasty number

a. $y(3)+x(0)$

b. $y(4)+x(0)$

c. $\frac{1}{2}y(3)$

d. $x(4)-x(0)-z(1)$

e. $2y(3)+x(2)+z(-3)$

20. Each of the following expression represents a cube number

a. $z(3)-z(-1)$

b. $y(5)-2z(-1)$

c. $x(5)+z(-2)$

d. $x(5)+y(5)-y(3)$

e. 2[x(5)+y(5)+x(4)+y(4)]

21. Each of the following expression represents a perfect square

a. $x(3)+z(1)$

b. $x(2)+z(2)$

c. $x(1)+y(2)$

d. $x(2)+y(0)$

e. $x(4)+x(0)$

f. $x(5)+y(5)+x(1)$

g. $x(5)+y(5)+x(4)+y(4)+3z(-3)$

h. $y(4)+x(3)+x(2)+z(-1)$

i. $2x(5)+z(1)$

j. $2y(5)+3z(3)$

Conclusion

In this paper, we have presented three different patterns of non- zero distinct integer solutions of ternary cubic Diophantine equation $x^2-xy+y^2=4z^3$ and relations between solutions and special numbers are also obtained. To conclude, one may search for other patterns of solutions and their corresponding properties.

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