ORIGINAL RESEARCH PAPER

Mathematics

On the Ternary Quadratic Equation $2y^2+xy=4z^2$.

KEY WORDS: Integral solutions, Ternary Quadratic.

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ABSTRACT

The ternary quadratic Diophantine equation is analyzed for its patterns of non zero distinct integral solutions.

Notations:

$t_{m,n}$ = Polygonal number of rank n with size m

Gno_n= Gnomic number of rank n

FN_n^4 = Four dimensional figurate number whose generating polygon is square.

Sqp_n = Square pyramidal number of rank n.

P_A^n = Pentagonal number of rank n.

Pr_n = Pronic number of rank n.

S_n= Star number of rank n.

SO_n= Stella octangular number of rank n

OH_n = Octahedral number.

J_n= Jacobthal number of rank n

j_n=jacobthallucas number of rank n

Obl_n = Oblong number of rank n.

PP_n = pentagonal pyramidal number of rank n.

Tet_n = Tetrahedral number of rank n.

Cp_n^6 = Centered pyramidal number of rank n with size m

Cp_n^{14} = Centered tetradecagonal pyramidal number of rank n

Introductions

Ternary quadratic equations are rich in variety [1-3]. For an extensive review of sizable literature and various problems, one may refer [4-19]. In this communication, we consider yet another interesting ternary quadratic equation, $2y^2 + xy = 4z^2$ and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions, special polygonal number, pyramidal number are presented.

II Method of Analysis:

The ternary quadratic equation to be solved in integers is

$$2y^2 + xy = 4z^2 \quad (1)$$

Now, introducing the linear transformation

$$x = u + v, \; y = u - v \quad (2)$$

in (1), which leads to

$$3u^2 + v^2 - 4uv = 4z^2 \quad (3)$$

$$u(3u - 4v) = (2z + v)(2z - v) \quad (4)$$

SET: 1

(3) can be written as $(v - 2u)^2 = u^2 + (2z)^2$

(i) Let $p = pq, u = p^2 - q^2$ and $v = 3p^2 - q^2$

Hence,

$$x(p,q) = 4p^2 - 2q^2$$

$$y(p,q) = -2p^2$$

$$z(p,q) = pq$$

(ii) Let $2z = p^2 - q^2,\ u = 2pq, v = p^2 + q^2 + 4pq$

By taking $p = 2P, q = 2Q, z = P - Q, u = 8PQ, v = 2P^2 + 2Q^2 + 16PQ$

Hence,

$$x(P,Q) = 2P^2 + 2Q^2 + 24PQ$$

$$y(P,Q) = 2P^2 + 2Q^2 + 8PQ$$

$$z(P,Q) = P - Q$$

SET: 2

Writing equation (3) as

$$\frac{u}{(2z+v)} = \frac{(2z-v)}{3u-4v} = \frac{A}{B} \quad (5)$$

Which is equivalent to the system of equations

$$Bu - Av - 2AZ = 0$$

$$3Au - (4A - B)v - 2Bz = 0$$

From which we get

$$u = -8A^2 + 4AB\ ;\ v = 2B^2 - 6A^2 \quad (6)$$

Using (6) in (2), we obtain the integer solutions to (1) as given below:

$$x = -14A^2 + 2B^2 + 4AB$$

$$y = -2A^2 - 2B^2 + 4AB$$

$$z = 3A^2 + B^2 - 4AB$$

Properties:

1) $x(A, 2A^2 - 1) + y(A, 2A^2 - 1) + 16obl_A - 8SO_A \equiv 0 (mod\ 16)$
2) $3y((B+1), B) + 2z((B+1), B) - 4Pr_B + t_{18,B} \equiv 0 (mod\ 7)$
3) $x(A, 7A^2 - 4) - 2z(A, 7A^2 - 4) + 5t_{4,2A} + 12CP_A^{14} = 0$
4) $z(A, 5) - 23t_{4,A} + 20obl_A - j_4 + 9 = 0$
5) $14z(B(B+1), B) + 3x(B(B+1), B) - t_{42,B} + 44P_B^5 \equiv 0 (mod\ 19)$
6) $y(A, (A+1)(A+2)) + x(A, (A+1)(A+2)) + 4t_{4,2A} - 8P_A^3 = 0$
7) $z(A, A(A+1) - Biq_A - 2CP_A^6 = 0$
8) $y(n, 1) + z(n, 1) + 1 = n^2$

It is observed that, by rewriting (4) suitably, one may arrive at the following three patterns of solution to (1)

A. PATTERN 1:

$$x(A,B) = -14A^2 + 2B^2 - 4AB$$

$$y(A,B) = -2A^2 - 2B^2 - 4AB$$

$$z(A,B) = -3A^2 - B^2 - 4AB$$

Properties:

1) $y\big(A,(A+1)(2A+1)\big) - 2z\big(A,(A+1)(2A+1)\big) - t_{10,A} - 24SqP_A \equiv 0(mod\ 3)$
2) $z(A^2 + A, 2A^2 - 1) + 15\mathrm{Biq}_A - 3\mathrm{CP}_A^{14} + 21\mathrm{cub}_A - 5\mathrm{t}_{4,A} \equiv 4(\mathrm{mod}\ 8)$
3) $x\big(A(A+1),(A+2)\big) + 2y\big(A(A+1),(A+2)\big) - 240\ FN_A^4 - 10t_{4,2A} + 40\ CP_A^6 - 72Tet_A = 0$
4) $x\big(A,A(A+1)\big) + y\big(A,A(A+1)\big) - 4t_{4,2A} - 8Pr_A = 0$
5) $x\big((A+1),(A+2)\big) + y\big((A+1),(A+2)\big) - S_A + 30obl_A + j_5 \equiv 2(mod\ 20)$
6) $y(A,1) + 2Pr_A - GNO_A \equiv 1(mod\ 4)$
7) $2z(A,A^2+1) + x(A,A^2+1) - 18\ \mathrm{OH}_A - 6\ \mathrm{obl}_A - 14\mathrm{t}_{4,A} = 0$
8) $x(A,A) + y(A,A)\ is\ a\ nasty\ number$

B. PATTERN 2:

$$x(A,B) = 2A^2 - 14B^2 - 4AB$$

$$y(A,B) = -2A^2 - 2B^2 - 4AB$$

$$z(A,B) = -A^2 - 3B^2 - 4AB$$

Properties:

1) $x(2A,A^2) + \mathrm{y}(2A,A^2) - 16t_{4,A^2} + 16\ CP_A^6 = 0$
2) $y(A^2,A+1) - 2z(A^2,A+1) - 8PP_A - 4\ obl_A \equiv 4(mod\ 4)$
3) $x\big(A,(A+1)(2A+1)\big) - 7y\big(A,(A+1)(2A+1)\big) - 144\ SqP_A - t_{34,A} \equiv 0(mod\ 15)$
4) $2z\big(A,(A+1)(A+2)\big) - 3y\big(A,(A+1)(A+2)\big) - 24\ P_A^3 - 48t_{3,A} + t_{42,A} \equiv 0(mod\ 43)$
5) $3x(A,7A^2-4) - 14\mathrm{z}(A,7A^2-4) - 132\ \mathrm{CP}_A^{14} - 40\ \mathrm{t}_{3,A} \equiv 0(\mathrm{mod}\ 20)$
6) $x(A,A) - y(A,A) + z(A,A) + 16Pr_A \equiv 0(mod\ 16)$
7) $y(A,5) + 2t_{4,A} \equiv 50(mod\ 20)$
8) $z(A,1) + y(A,1) + 6t_{3,A} + j_2 \equiv 0(mod\ 5)$

C. PATTERN 3:

$$x(A,B) = 2A^2 - 14B^2 + 4AB$$

$$y(A,B) = -2A^2 - 2B^2 + 4AB$$

$$z(A,B) = A^2 + 3B^2 - 4AB$$

Properties:

1) $x\big(A(A+1),(A+2)) + y(A(A+1),(A+2)\big) + 16Gno_A + 14t_{4,2A} - 48Tet_A \equiv 0(mod\ 80)$
2) $2z\big((A+1),A^2\big) + y\big((A+1),A^2\big) - 4Biq_A + 8PP_A = 0$
3) $x((A-1),A^2) - 7\mathrm{y}\big((A-1),A^2\big) + 36\ OH_A - 40\ Obl_A - 9J_3 - j_2 \equiv 1(mod\ 60)$
4) $z(A^2,A) - 12FN_A^4 - 2t_{4,A} - 4CP_A^6 = 0$
5) $3x(A,2A^2+1) + 14z(A,2A^2+1) - 5t_{4,2A} + 44\ SO_A \equiv 0(mod\ 88)$
6) $x(A,2) - y(A,2) + 2z(A,2) - S_A + 3J_4 + j_3 \equiv 3(mod\ 18)$
7) $x(A,1) - 2Pr_A - GNo_A + 3J_3 \equiv 0(mod\ 4)$
8) $y\big((A^2+A),2A^2-1\big) + 24\mathrm{FN}_A^4 - 4\mathrm{CP}_A^6 \equiv 2(\mathrm{mod}\ 4)$

III. CONCLUSION

One may search for other Choices of solutions and their corresponding properties.

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