ORIGINAL RESEARCH PAPER

Mathematics

A STUDY ON VISIBILITY GRAPH TYPES AND ITS APPLICATION IN ART GALLERY THEOREMS

KEY WORDS: Visibility graphs, Art gallery problem, polygon with holes, Exterior visibility.

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ABSTRACT

In computational geometry and robot motion planning, a visibility graph is a graph of intervisible locations, typically for a set of points and obstacles in the Euclidean plane. Visibility graphs may also be used to calculate the placement of radio antennas, or as a tool used within architecture and urban planning through visibility graph analysis. This is a brief survey on the visibility graphs on polygons, orthogonal polygons, and polygons with holes. Here are some basic definitions and applications of visibility graphs.

1.Introduction

In a visibility graph, each node in the graph represents a point location, and each edge represents a visible connection between them. That is, if the line segment connecting two locations does not pass through any obstacle, an edge is drawn between them in the graph. Lozano-Perez & Wesley (1979) attribute the visibility graph method for Euclidean shortest paths to research in 1969 by Nils Nilsson on motion planning for Shakey the robot, and also cite a 1973 description of this method by Russian mathematicians M. B. Ignat'yev, F. M. Kulakov, and A. M. Pokrovskiy. ElGindy has pioneered their investigation in his thesis (ElGindy 1985). He obtained a specialized result by restricting the class of graphs to maximal outerplanar graphs. Although this result is very restricted, it is the most general obtained to date.Visibility graphs may also be used to calculate the placement of radio antennas, or as a tool used within architecture and urban planning through visibility graph analysis. The art gallery problem is the problem of finding a small set of points such that all other non-obstacle points are visible from this set.

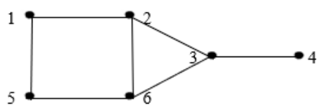
2.Preliminaries [4,14,19]

The nodes of a visibility graph correspond to geometric components, such as vertices or edges, and two nodes are connected by an arc of the graph if the components can "see" one another, perhaps under some restricted form of visibility.[19] The canonical example is the

vertex visibility graph of a polygon: its nodes correspond to the vertices of a polygon, and its arcs to lines of visibility between vertices in the interior or along the boundary of the polygon. A polygon of n vertices will sometimes be called an n-gon.

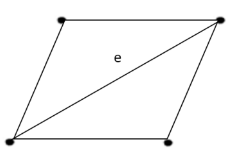
Definition 2.1. A graph G consists of a pair (V(G), X(G)), where V(G) is a non-empty finite set whose elements are called points or vertices and X(G) is a set of unordered pairs of distinct elements of V(G). The elements of X(G) are called lines or edges of a graph G. If $x = \{u, v\} \in X(G)$, the line x is said to join u and v. We write x= uv and we say that the points u and v are adjacent.

Consider a graph



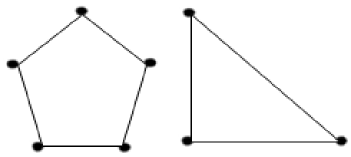
Here V= {1, 2, 3, 4, 5, 6} are the vertices of this graph and X= {{1,2}, {1,5}, {2,3}, {2,6}, {3,4}, {3,6}, {5,6}} are the lines of this graph and 1 is adjacent to 2 and 5.

Definition 2.2 A diagonal is a line segment joining two non-consecutive vertices of a polygon or polyhedron.
Consider a graph



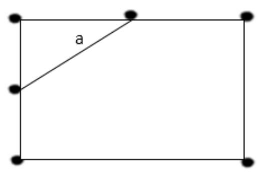
Here 'e' is a diagonal.

Definition 2.3 A Simple polygon is defined as a flat shape consisting of straight non-intersecting line segments or "sides" that are joined pair wise to form a closed path.



Definition 2.4 The arcs on the exterior face will be called exterior arcs; all other are interior arcs.

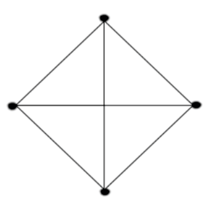
Consider a graph



Here a is the interior arcs, while the others are exterior arcs.

Definition 2.5 An assignment of colours to the vertices of a graph so that no two adjacent vertices get the same color is called as vertex coloring.

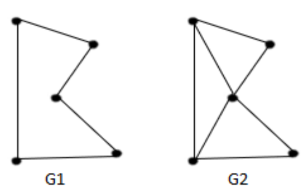
Consider a graph



It is a 4-colouring graph

Definition 2.6 A graph is called maximal planar if no line can be added to it without losing planarity. In a maximal planar graph, each face is a triangle and such a graph is sometimes called a triangulated graph.

Consider the graph



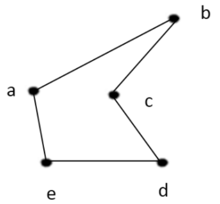
The graph G1 is triangulated by diagonal and thus G2 is formed.

3. Comparative study of visibility graphs [1,6,7,11,21,23]

3.1 Vertex visibility graph

The nodes of a visibility graph correspond to geometric components, such as vertices, and two vertices are connected by an undirected edge of the graph if the components can "see" one another, perhaps under some restricted form of visibility. This is known as vertex visibility graph. It is denoted as G_V.

Consider a graph



Here vertex 'a' is visible to vertex 'b' and 'e'.
Vertex 'b' is visible to vertex 'a' and 'c'.
Vertex 'c' is visible to vertex 'b' and 'd'.
Vertex 'd' is visible to vertex 'c' and 'e'.
Vertex 'e' is visible to vertex 'a' and 'd'.

Theorem 1: Every maximum outer planar graph G is a vertex visibility graph of a monotone polygon.

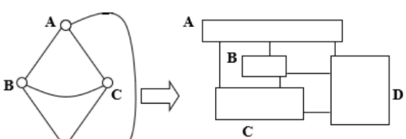
3.2 Edge visibility graph [2,20]

The edge visibility graph GE of a polygon P is to have a node for each edge of P, and an arc (ei, ej) GE if and only if ei sees ej, i.e., if and only if there is a point x on the (open) edge ei and a point y on the (open) edge ej such that x sees y.

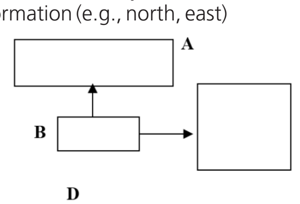
3.3 Rectangle visibility graph [5,19]

Let R = { Ri} be a collection of pairwise disjoint closed rectangles in the plane. Two rectangles Ri and Rj are called visible if there is a closed non-degenrate rectangular region Bij (called a band of visibility) such that one side of Bij is contained in a side of Rj, and Bij does not intersect the interior of any rectangle in R. The visibility graph of R is the graph of the visibility relation on vertex-set R; a collection of rectangles and its visibility graph. A graph is called a rectangle visibility graph or RVG, if it is the visibility graph of some collection R of rectangles.

Let graph G = (V, E) then G be realized as the visibility graph of rectangles

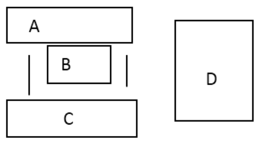


- horizontal vs. vertical visibility
- Direction information (e.g., north, east)

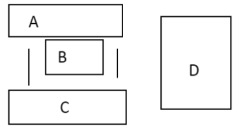


B sees A vertically, to the north; B sees D horizontally, to the east

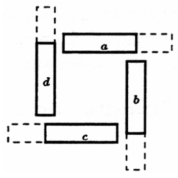
- multiple edges



A sees C on both sides of B



Theorem 2: If {a,b,c,d}is a cyclic four-way in a graph G , and G is laid out in a way that respects its colouring, then the rectangles {a,b,c,d} must be laid out as shown in the figure

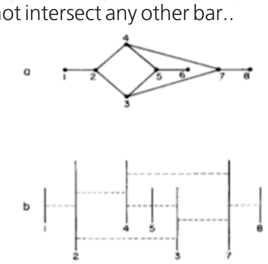


Theorem 3: For 1 ≤ k ≤ 4, every k-tree is a non-collinear rectangle-visibility graphs.

Theorem 4: Every partial 2-tree is a non-collinear rectangle-visibility graph.

3.4 Bar visibility graphs [8,19]

In a bar visibility graph (or just a bar graph), the nodes represent vertical line segments, and two nodes are connected by an arc if and only if their two vertical bars A and B can see each other horizontally and non-degenerately. More precisely, there must exist a non-zero height rectangle bounded by A and B to the right and left that does not intersect any other bar..



Theorem 5: Every 1-tree and 2-tree is a non-collinear bar-visibility graph.

Theorem 6: Every partial 1-tree (forest) is a non-collinear bar-visibility graph.

4. Comparative study on Art gallery problems [12,13,15, 16,17,19,22]

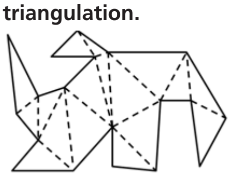
Chvatal constructed the first proof of the art gallery problem in 1975. It was very elaborate and used induction. In 1978 Steve Fisk constructed a much simpler proof based on dividing a polygon into triangles using diagonals. So, Fisk's method is used for proving or solving the problem. The art gallery problem is the problem of finding a small set of points such that all other non-obstacle points are visible from this set.

4.1 The problem:

What is the art gallery problem?
For instance, if one own an art gallery and want to place security cameras so that the entire gallery will be safe from thieves.

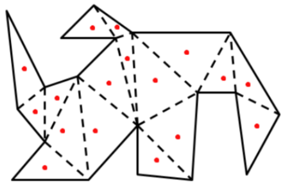
a. Where the cameras should be placed?
b. What is the minimum number of cameras needed to keep the art collection safe?

Some notes on triangulation of the problem: Decomposition of a polygon into triangles by a maximal set of non-intersecting diagonals is known as **triangulation.**



To make things easier, decompose a polygon into pieces that are easy to guard. Draw diagonals (an open line segment that connects two vertices and lie in the interior of the polygon) between pair of vertices.

Guard the polygon by placing a camera in every triangle.



Lemma 1: Every simple polygon has a triangulation. Any triangulation of a simple polygon with n vertices consists of exactly n – 2 triangles.

Notes: Lemma 1 implies that n – 2 cameras can guard the simple polygon. A camera on diagonal guards two triangles. Therefore, cameras can be reduced to roughly n/2. A vertex is adjacent to many triangles. So, placing cameras at vertices can do even better.

Theorem 7:(Chvatal's Art Gallery Theorem 1975)n 3 guardsareoccasionally necessary and always sufficient to cover a polygon with n vertices.

4.2 Polygon partitions

Theorem 8: (Triangulation Theorem). A polygon of n vertices may be partitioned into n-2 triangles by the addition of n-3 internal diagonals

Theorem 9: [Meister's Two Ears Theorem 1975]. Every polygon of n ≥ 4 vertices has at least two non-overlapping ears.

Theorem 10: [O'Rourke 1982]. r guards are occasionally necessary and always sufficient to see the interior of a simple n-gon of r ≥ 1 reflex vertices.

4.3 Orthogonal polygons

The orthogonal art gallery theorem was first formulated and proved by Kahn, Klawe, and Kleitman in 1980 (Kahn et al. 1983). It states that $\lfloor \frac{n}{4} \rfloor$ guards are occasionally necessary and always sufficient to see the interior of an orthogonal art gallery room.

Theorem 11: [Kahn, Klawe, and Kleitman 1980]. Every orthogonal polygon P (with or without holes) is convexly quadrilateralizable.

Theorem 12: [Lubiw 1985]. Any 1-orthogonal polygon P is convexly quadrilateralizable.

Theorem 13:$\lfloor \frac{r}{2} \rfloor$+ 1 guards are necessary and sufficient to cover the interior of an orthogonal polygon of r reflex vertices.

4.4 Mobile guards [9,18,19,22]

Each guard is permitted to "patrol" an interior line segment. Let S be a line segment completely contained in the closed polygonal region P: S⊆P. Then x ϵ P is said to be seen by S, or is covered by S, if there is a point y ϵ S such that the line segment xy ⊆ P. Thus, x is covered by the guard if x is visible from some point along the guard's patrol path.

Guard / Shape	Stationary	Mobile
General	$\lfloor \frac{n}{3} \rfloor$	$\lfloor \frac{n}{4} \rfloor$
Orthogonal	$\lfloor \frac{n}{4} \rfloor$	$\lfloor (3n+4)/16 \rfloor$

Theorem 14: 1 [O'Rourke 1983]. Every triangulation graph T of a polygon of n ≥ 4 vertices can be dominated by [n/4] combinatorialdiagonal guards.

Theorem 15: [Aggarwal 1984].[((3q+5)) 8]= [((3n+4))] /16mobile guards are sufficient to cover any orthogonalpolygon P of q quadrilaterals and n vertices.

4.5 Exterior Visibility [3,19]

Derick Wood and Joseph Malkelvitch independently posed two interesting variants of the original Art Gallery Problem, which Wood dubbed The Fortress Problem and The Prison Yard Problem

Fortress problem

How many vertex guards are needed to see the exterior of a polygon of n vertices? An exterior point v is seen by a guard at vertex z if and only if the segment zy does not intersect the interior of the polygon.

Theorem 16: [O'Rourke and Wood 1983]. $\lceil \frac{n}{2} \rceil$ vertex guards are necessary and sufficient to see the exterior of a polygon of n vertices.

Theorem 17: [Aggarwal 1983]$\lceil \frac{n}{4} \rceil$+ 1 vertex guards arenecessary and sufficient to see the exterior of an orthogonal polygon of n vertices.

Prison yard problem

How many vertex guards are needed to simultaneously see the exterior and interior of a polygon P of n vertices? An interior point x is seen by a guard at vertex z if the segment zx does not intersect the exterior of P, and an exterior point y is seen by z if zv does not intersect the interior of P.

Theorem 18: [O'Rourke 1983]. [7n/16]+ 5 vertex guards aresufficient to see both the interior and exterior of a simple orthogonal polygon.

Problem	Techniques	Guards
Fortress **General** **Orthogonal**	Triangulation, 3-coloring L-shaped partition	$\lceil n/2 \rceil$ $\lceil n/4 \rceil + 1$
Prison Yard **General**	Exterior Triangulation, 4-coloring Triangulation, 4-coloring Exterior, triang., 3-coloring	$\lceil n/2 \rceil + r$ $\lfloor (n + \lceil h/2 \rceil)/2 \rfloor$ $\lfloor 2n/3 \rfloor$ $\lfloor 2n/3 \rfloor + 1$
Orthogonal	Exterior, quadrilateralization, 4-coloring	$\lfloor 7n/16 \rfloor + 5$

5. Conclusion

Visibility graph has so many applications in the field of Computer Science and Mathematics. Some types of visibility graphs are discussed in this paper. Visibility graphs may be used to find Euclidean shortest paths among a set of polygonal obstacles in the plane: the shortest path between two obstacles follows straight line segments except at the vertices of the obstacles, where it may turn, so the Euclidean shortest path is the shortest path in a visibility graph that has as its nodes the start and destination points and the vertices of the obstacles. Therefore, the Euclidean shortest path problem may be decomposed into two simpler sub problems: constructing the visibility graph and applying a shortest path algorithm such as Dijkstra's algorithm to the graph.

For planning the motion of a robot that has non-negligible size compared to the obstacles, a similar approach may be used after expanding the obstacles to compensate for the size of the robot. Lozano-Perez & Wesley (1979) attribute the visibility graph method for Euclidean shortest paths to research in 1969 by Nils Nilsson on motion planning for Shakey the robot, and also cite a 1973 description of this method by Russian mathematicians M. B. Ignat'yev, F. M. Kulakov, and A. M. Pokrovskiy. Visibility graphs may also be used to calculate the placement of radio antennas, or as a tool used within architecture and urban planning through visibility graph analysis. Further study on the visibility graphs may reveal more analogous results of these kind and will be discussed in the forthcoming papers.

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