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Mathematics

TOPOLOGICAL PROPERTIES IN REGULAR GENERALIZED M-FUZZY METRIC SPACE

KEY WORDS: Neighborhood, balanced set, convex set.

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ABSTRACT

The aim of this paper is to introduce regular generalized M-fuzzy metric space and study the properties of topology induced by this metric.

1. INTRODUCTION

In 1965, the concept of fuzzy sets was introduced by Zadeh [12]. Since then many authors have expansively developed the theory of fuzzy sets and applications. Especially Kaleva and Seikkala [4], Sedghi and Shobe [6], Kramosil and Michalek [5] have introduced the concepts of fuzzy metric spaces in different ways. We introduced generalized *M*-fuzzy mertic space [7]. The main objective of this paper is to study the properties of topology induced by this metric space.

2. Preliminaries

Definition 2.1. [7]

A 3-tuple (X, *M*,*) is called a generalized *M*-fuzzy metric space if X is an arbitrary nonempty set,* is a continuous t-norm and $M : X^n$ $(0,\infty) \to [0, 1]$, n 3 satisfying the following conditions, for each x_1, $x_2, \ldots, x_n, x_n \in X$ and t, s > 0.

(I) $M(x_1, x_2, \ldots, x_n, t) > 0$
(ii) $M(x_1, x_2, \ldots, x_n, t) = 1$ for all t > 0 if and only if $x_1 = x_2 = \ldots = x_n$.
(iii) $M(x_1, x_2, \ldots, x_n, t) = M(p(x_1, x_2, \ldots, x_n), t)$ where p is a permutation function.
(iv) $M(x_1, x_2, \ldots, x_n, t+s)$ $M(x_1, x_2, \ldots, x_{n1}, x_n, t) * M(x_n, x_n, \ldots, x_n, s)$
(v) $M(x_1, x_2, \ldots, x_n, .) : (0, \infty) \to [0, 1]$ is continuous.
(vi) $M(x_1, x_2, \ldots, x_n, t) \to 1$ as $t \to \infty$.

Example 2.2. [7]

Let (X, *M*, *) be a modified fuzzy metric as in [3] which satisfy the additional condition $M(x, y, t) \to 1$ as $t \to \infty$. Define $M(x_1, x_2, \ldots, x_n, t) = M(x_1, x_2, t) * M(x_2, x_3, t) * \ldots *$

$M(x_{n1}, x_n, t)$ for every $x_1, x_2, \ldots, x_n \in X$. Then (X, *M*, *) is a generalized *M*-fuzzy metric space.

Example 2.3. [9]

Consider X = R. Let * be the product norm defined by a * b = ab.

Define $M(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + |x_1 - x_2| + \cdots + |x_{n-1} - x_n|}$ Then (X, *M*, *) is a generalized *M*-fuzzy metric space.

Example 2.4. [7]

Let (X, *M*, *) be a generalized *M*-fuzzy metric space and let $x \in X$, 0< r < 1, t > 0 we define $B_M(x, r, t) = \{y \; X / M(y, \ldots, y, x, t) > 1r\}$ called open ball.

Definition 2.5. [7]

Let (X, *M*, *) be a generalized *M*-fuzzy metric space. A subset A of X is said to be open if for each $x \in A$ there is a 0 < r < 1, t > 0 such that $B_M(x, r, t) \subseteq A$.

Theorem 2.6. [7]

Let (X, *M*, *) be a generalized *M*-fuzzy metric space. Then every open ball is an open set.

Definition 2.7. [7]

A topology on a set X is a collection of subsets of X having the following properties

(i) ϕ and X are in τ
(ii) The union of the elements of any subcollection of τ is in τ.
(iii) The intersection of the elements of any finite subcollection of is in .

A set X for which a topology has been specified is called a topological space.

Theorem 2.8. [7]

Let (X, *M*, *) be a generalized *M*-fuzzy metric space. Then $\tau_M = \{A /$ A is open in X} is a topology on X called topology induced by *M*. We call (X, τ_M) is a topological space induced by *M*.

Definition 2.9. [1]

A topological space (X,τ) is called a Hausdorff space if for each pair x_1, x_2, of distinct points of X, there exists neighborhoods U_1 and U_2 of x_1 and x_2 respectively that are disjoint.

Theorem 2.10. [7] Every generalized *M*-fuzzy metric space is a Hausdorff space.

Definition 2.11. [7]

Let (X, *M*, *) be a generalized *M*-fuzzy metric space and let A and B are disjoint closed subsets of X. X is said to be normal space if there exists disjoint open sets U and V such that $A \subseteq U$, $B \subseteq V$.

Theorem 2.12. [1]Every generalized *M*-fuzzy metric space is a normal space.

3. Topology Induced by Regular Generalized *M*-fuzzy Metric Space

Here we introduce regular generalized *M*-fuzzy metric space and study the properties of topology induced by this metric.

Definition 3.1.

A generalized *M*-fuzzy metric space (X, *M*, *) is said to be a regular generalized *M*-fuzzy metric space provided

(i) X is a real or complex vector space
(ii) $M(x_1+a, x_2+a, \ldots, x_n+a, t) = M(x_1, x_2, \ldots, x_n, t)$
(iii) $M(x_1, x_2, \ldots, x_n, |k|t) = M(\frac{x_1}{k}, \ldots, \frac{x_n}{k}, t)$.

Now we give an example which ensure the existence of a regular generalized *M*-fuzzy metric space.

Example 3.2.

Let (X, *M*, *) be a generalized *M*-fuzzy metric space. Let X = R and define $M(x_1, x_2, \ldots, x_n, t) = e^{\frac{-|x_1 - x_2|}{t}} \ldots e^{\frac{-|x_{n-1} - x_n|}{t}}$

Then (X, M, *) is a regular generalized M-fuzzy metric space.

Theorem 3.3.
Let (X, M, *) be a regular generalized M-fuzzy metric space.

Then for all x, y $\in$ X, $0 < r < 1$ and $t > 0$. We have $y + B_M(x, r, t) = B_M(x{+}y, r, t)$.

Proof.

By our definition $B_M(x, r, t) = \{z \in X; M(z, x, \ldots, x, t) > 1r\}$.

Now $y + B_M(x, r, t) = \{y + z; M(z, x, \ldots, x, t) > 1r\}$.

As per our metric property, $M(z, x, \ldots, x, t) > 1r$ if and only if $M(z{+}y, x{+}y, \ldots, x{+}y, t) > 1r$ if and only if $M(z{+}y, x{+}y, \ldots, x{+}y, t) > 1r$.

This means that $z + y \in B_M(x + y, r, t)$ and so $y + B_M(x, r, t) = B_M(x + y, r, t)$.

Theorem 3.4.
Let (X, M, *) be a regular generalized M-fuzzy metric space.Then the following hold

(I) If A is a open set in X and y $\in$ X then y + A is a open set.
(ii) If A is a open set in X and B $\subseteq$ X then A + B is a open set.

Proof.

(I) Suppose A is a open set. Then for all x $\in$ A there exists $0 < r < 1$ and $t > 0$ such that $B_M(x, r, t) \subseteq A$. Since $B_M(x + y, r, t) = B_M(x, r, t) + y$, $B_M(x + y, r, t) \subseteq y + A$ and so y + A is a open set.
(ii) It is seen that $A + B = \{x + y; x \in A, y \in B\}$. For any y B. Since B is a open set, there is a $0 < r < 1$ and $t > 0$ such that $B_M(y, r, t) \subseteq B$.

Now $x + B_M(y, r, t) = B_M(x + y, r, t) \subseteq A + B$ for all x $\in$ A. Hence for any $x + y \in A + B$, the open set $B_M(x + y, r, t) \subseteq A + B$. This means that A + B is a open set.

Theorem 3.5.
Let (X, M, *) be a regular generalized M-fuzzy metric space. Then $B_M(0, r, |k|t) = kB_M(0, r, t)$ for all scalar k.

Proof.
We have

$B_M(0, r, |k|\, t) = \{z \ \ X : M(0, z, \ldots, z, |kt|) > 1r\}$

$= \{z \in X : \mathcal{M}(0, \frac{z}{k}, \ldots, \frac{z}{k}, t) > 1{-}r\}$

$= k \{\frac{z}{k} \in X : \mathcal{M}(0, \frac{z}{k}, \ldots, \frac{z}{k}, t) > 1{-}r\}$

$= k \{y \in X : \mathcal{M}(0, y, \ldots, y, t) > 1{-}r\}$

$= k\, B_{\mathcal{M}}(0, r, t)$

Hence $B_M(0, r, |k|\, t) = k\, B_M(0, r, t)$.

Theorem 3.6.
Let (X, M, *) be a regular generalized M-fuzzy metric space and x $\in$ X. Then $x + B_M(0, r, |k|\, t) = k\, B_M(\frac{x}{k}, r, t)$.

Proof.

By the above theorem

$x + B_M(0, r, t) = k\, [\frac{x}{k} + B_{\mathcal{M}}(0,r,t)]$

$= k\, B_M(\frac{x}{k}, r, t)$

Definition 3.7.
Let (X, M, *) be a regular generalized M-fuzzy metric space and A $\subseteq$ X. Then A is called a neighborhood of a point x $\in$ X if there exist $B \in \tau_M$ such that $x \in B \subseteq A$.

Theorem 3.8.
Let (X, M, *) be a regular generalized M-fuzzy metric space. Let A be a subset of X. Then A is a neighborhood of x if and only if $B_M(x, r, t) \subset A$ for some $0 < r < 1$, $t > 0$.

Proof.

Suppose A is a neighborhood of x. Then there is a $0 < r < 1$ and $t>0$ such that $B_M(x, r, t) \subseteq A$.

Conversely, suppose $B_M(x, r, t) \subset A$. Since $x \in B_M(x, r, t) \subset A$, A is a neighborhood of x.

Theorem 3.9.
Let (X, M, *) be a regular generalized M-fuzzy metric space. If A is a neighborhood of x then kA is a neighborhood of kx for all scalar k.

Proof.

Let A be a neighborhood of a point x $\in$ X.
By definition, there is a $0 < r < 1$ and $t > 0$ such that $B_M(x, r, t) \subseteq A$.
This implies that $x + B_M(0, r, t)$ A and so $kx + B_M(0, r, |k|\, t) \subseteq kA$.
i.e., $B_M(kx, r, |k|\, t) \subseteq kA$.
Hence kA is a neighborhood of kx.

Theorem 3.10.
Let (X, M, *) be a regular generalized M-fuzzy metric space. If A is a neighborhood of a point x_0. Then x_0 + A is a neighborhood of origin 0.

Proof.

Since A is a neighborhood of x_0,
By definition there is a $0 < r < 1$ and $t > 0$ such that $B_M(x_0, r, t) \subset A$.
This implies that $x_0 + B_M(0, r, t) \subseteq A$ and so $B_M(0, r, t) \subseteq x_0 + A$
Therefore $x_0 + A$ is a neighborhood of origin

Theorem 3.11. Let (X, M, *) be a regular generalized M-fuzzy metric space. Let A be the neighborhood of origin 0 and $x_0 \in X$. Then $x_0 + A$ is a neighborhood of x_0.

Proof.

Since A is a neighborhood of origin
By definition, there is a $0 < r < 1$ and $t > 0$ such that $B_M(0, r, t) \subset A$.
This implies that $x_0 + B_M(0, r, t) \subseteq x_0 + A$.
This means that $B_M(0, r, t) \subseteq x_0 + A$
Therefore $x_0 + A$ is a neighborhood of x_0.

Theorem 3.12.
Let (X, M, *) be a regular generalized M-fuzzy metric space. Then $B_M(0, r, t)$ is a balanced set.

Proof.
Let k be a scalar such that $|k| \leq 1$
Now $k\, B_M(0, r, t) = B_M(0, r, |k|\, t) \subseteq B_M(0, r, t)$
Therefore $B_M(0, r, t)$ is a balanced set.

Theorem 3.13.
Let (X, M, *) be a regular generalized M-fuzzy metric space and x $\in$ X. Then $B_M(x, r, t)$ is a convex set.

Proof.

Let $x_1, x_2 \in X$, $0 < r < 1$ and $t_1, t_2 > 0$ we have
$B_M(x_1, r, t_1) + B_M(x_2, r, t_2) = x_1 + B_M(0, r, t_1) + x_2 + B_M(0, r, t_2)$
Suppose $t_2 = k\, t_1$ for some k
$x_1 + B_M(0, r, t_1) + x_2 + B_M(0, r, k\, t_1)$

$= x_1 + x_2 + B_M(0, r, t_1) + k\, B_M(0, r, t_1)$
$= x_1 + x_2 + B_M(0, r, (1{+}k)\, t_1)$
$= x_1 + x_2 + B_M(0, r, t_1 + k\, t_1)$
$= x_1 + x_2 + B_M(0, r, t_1 + t_2)$
$= B_M(x_1 + x_2, r, t_1 + t_2)$

Hence $B_M(x_1, r, t_1) + B_M(x_2, r, t_2) = B_M(x_1 + x_2, r, t_1 + t_2)$
Let $k \in [0, 1]$. Then

$$\begin{aligned} k\,B_M(0, r, t) + (1k)\,B_M(0, r, t) &= B_M(0, r, k\,t) + B_M(0, r, (1k)\,t) \\ &= B_M(0, r, k\,t + t\ \ k\,t) \\ &= B_M(0, r, t) \end{aligned}$$

Hence $B_M(0, r, t)$ is a convex set.
Now for any $x \in X$.

$$\begin{aligned} k\,B_M(x, r, t) + (1k)\,B_M(x, r, t) &+ k[x + B_M(0, r, t)] + (1k)\,[x + B_M(0, r, t)] \\ &= kx + (1k)\,x + k\,B_M(0, r, t) + (1k)\,B_M(0, r, t) \\ &= x + B_M(0, r, t) \\ &= B_M(x, r, t) \end{aligned}$$

Therefore $B_M(x, r, t)$ is a convex set.

Theorem 3.14. Let (X, *M*, *) be a regular generalized *M*-fuzzy metric space. Let A be an open set of X then k A is also a open set in X for all scalars k.

Proof.
Now $k\,A = \{k\,x : x \in A\}$. Let $x\ \ A$.

By definition, there is a $0 < r < 1$ and $t > 0$ such that $B_M(x, r, t) \subset A$.
Now $k\,B_M(x, r, t) = B_M(k\,x, r, |k|\,t) \subseteq k\,A$.
Hence k A is a open set in X all scalar k.

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