ORIGINAL RESEARCH PAPER

Mathematics

ON THE HOMOGENEOUS BI-QUADRATIC EQUATION $X^4 - Y^4 = 26(2Z^2 - 2W^2)T^2$ WITH FIVE UNKNOWNS

KEY WORDS: Bi-Quadratic equation, Integral solutions, Special polygonal numbers, Pyramidal numbers.

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ABSTRACT

The Bi-quadratic equation with five unknowns given by $X^4 - Y^4 = 26(2Z^2 - 2W^2)T^2$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

INTRODUCTION

Bi-quadratic Diophantine Equations(homogeneous and non-homogeneous) have aroused the interest of numerous mathematicians since ambiguity as can be seen from [1 – 7]. In the context one may refer [8 – 17] for varieties of problems on the Diophantine equations with two, three and four variables. This communication concerns with the problems of determining non-zero integral solutions of yet another bi-quadratic equation in five unknowns represented by $X^4 - Y^4 = 26(2Z^2 - 2W^2)$. A few interesting relations between the solutions and special polygonal numbers are presented.

Notations used:

* $T_{m,n}$ – Polygonal number of rank n with size m.
* Pr_n – Pronic number of rank n.
* SO_n – Stella Octangular number of rank n.
* Obl_n – Oblong number of rank n.
* OH_n – Octahedral number of rank n.
* GnO_n – Gnomic number of rank n.
* PP_n – Pentagonal Pyramidal number of rank n.
* S_n – Star number of rank n.
* Ky_n – Kynea number of rank n.
* CP_n^{14} – Centered tetra decagonal pyramidal number of rank n.
* CH_n – Centered Hexagonal number of rank n.
* CP_n – Centered pentagonal number of rank n.
* $4DF_n$ – Four Dimensional figurate number where generating polygon is a square.
* j_n – Jacobthal Lucas number of rank n.
* J_n – Jacobthal number of rank n.

METHOD OF ANALYSIS

The Diophantine equation representing the bi-quadratic equation with five unknowns under consideration is

$$x^4 - y^4 = 26(2z^2 - 2w^2)T^2 \qquad (1)$$

The substitution of the linear transformations

$x = 2u + 2v;\ y = 2u - 2v;\ z = 4u + 4v;\ w = 4u - 4v$ (2) In (1) leads to $u^2 + v^2 = 26T^2$ (3)

Different patterns of solutions of (1) are presented below:

Pattern – 1

Equation (3) can be written as

$$\frac{u+T}{5T+v} = \frac{5T-v}{u-T} = \frac{a}{b} \qquad (4)$$

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From equation (4), we get the values of u, v and T

$$u = -a^2 + b^2 - 10ab$$

$$v = 5a^2 - 5b^2 - 2ab$$

$$T = -(a^2 + b^2)$$

Hence in view of (2) the corresponding solutions of (1) are

$$x = x(a,b) = 8a^2 - 8b^2 - 24ab$$

$$y = y(a,b) = -12a^2 + 12b^2 - 16ab$$

$$z = z(a,b) = 16a^2 - 16b^2 - 48ab$$

$$w = w(a,b) = -24a^2 + 24b^2 - 32ab$$

$$T = -(a^2 + b^2)$$

Properties :

1. $3x(a(a+1), 2a+1) + w(a(a+1), 2a+1) + 624P_a^4 = 0$
2. $x(a, 5a-3) + z(a, 5a-3) - w(a, 5a-3) + 208T_{7,a} = 0$
3. $z(a(a+1), a) - 16T(a(a+1), a) - 384FN_a^4 - 4T_{4,4a} + 64CP_a^6 + 96P_a^5 = 0$
4. $x(b(b+1), 2b+1) - 4T(b(b+1), 2b+1)$

$- y(b(b+1), 2b+1) - 32Pr_b + 240SP_b - j_3 \equiv 0 \pmod 1$

5. $x(7b^2 - 4, b) + 8T(7b^2 - 4, b) + t_{34,b} + 72CP_b^{14} \equiv 0 \pmod{15}$

6. $y(a^2,\ a+1) + 12T_{4,a^2} - 2\,Obl_a - GnO_a - 3 = 0$

7. $x(2a^2 - 1, a) - 384FN_a^4 + t_{18,a} \equiv 8 \pmod 7$

8. $z(b, 2b^2 + 1) + 4T_{4,4b^2} + 12T_{4,2b} + j_4 + 144\,OH_b - 1 = 0$

9. $w(b(b+1), 5b-2) - 288FN_a^4 + 96P_b^5 - 6T_{4,10b} + 192HP_b \equiv 96 (mod\ 480)$

10. $T(3a^2 - 1, a^2) + 10T_{4,a^2} - 6T_{4,a} - 1 = 0$

11. Each of the following expressions represents a nasty number

    (i) $x(a,a)$ & $z(a,a)$

    (ii) $2y(a,a)$

    (iii) $6T(a,a)$ & $2w(a,a)$

12. $y(a,a)$ is a perfect square.

Pattern – 2

Assume $T = T(a,b) = a^2 + b^2$ where a and b are non-zero distinct integers (5)

write 26 as $26 = (1+5i)(1-5i)$ (6)

using (5) & (6) in (3), and employing the method of factorization, define

$$u + iv = (1+5i) \qquad (a+ib)^2$$

Equating the real and imaginary parts, we get

$$u = u(a,b) = a^2 - b^2 - 10ab$$

$$v = v(a,b) = 5a^2 - 5b^2 + 2ab$$

Hence in view of (2) the corresponding solutions of (1) are

$$x = x(a,b) = 12a^2 - 12b^2 - 16ab$$

$$y = y(a,b) = -8a^2 + 8b^2 - 24ab$$

$$z = z(a,b) = 24a^2 - 24b^2 - 32ab$$

$$w = w(a,b) = -16a^2 + 16b^2 - 48ab$$

Properties :

1. $x(a,2a^2-1) + y(a,2a^2-1) - w(a,2a^2-1) - 8SO_a$
$+ 960(4FD_a) - 20Pr_a + CP_3^6 - Ky_1 \equiv 0(mod\ 20)$
2. $x(7a^2-4,a) + y(7a^2-4,a) + T(7a^2-4,a) - 120CP_a^{14} - 2940FN_a^4 + T_{76,a} + Ky_3 - J_2 \equiv 0(mod\ 37)$
3. $3y(a(a+1),5a-2) + z(a(a+1),5a-2) + 624HP_a = 0$
4. $x(1,b) - 12T(1,b) + 48T_{3,b} - 8Pr_b + 8T_{4,b} = 0$
5. $y(a,2a^2+1) + 72\,OH_a - 8T_{4,2a^2} + CP_2^6 - 12SO_a \equiv 0(mod\ 12)$
6. $z(a(a+1),2a+1) + 192SqP_a - 6T_{4,2a^2} - 2T_{13,a} + 83\,Obl_a - 48CP_a^6 \equiv 0(mod\ 5)$
7. $z(b,b+1) + w(b,b+1) - 16T(b,b+1) + 80Pr_b + 32T_{3,b} \equiv 24(mod\ 32)$
8. $w(b,11b-9) - 2y(b,11b-9) - w(b,11b-9) + 1915\,Obl_b - 1296GnO_b - 48T_{13,b} + 2T_{7,b}\ 0(mod\ 2488)$
9. $2y(a(a+1),a+2) + 16T(a(a+1),a+2) + 288P_A^3 - T_{66,a} \equiv 32(mod\ 98)$
10. $x(b^2,b+1) + w(b^2,b+1) + 48(4DF_b) + 128PP_b - 20T_{3,b} - J_6 + Ky_1 \equiv 0(mod\ 18)$
11. Each of the following expressions represents a nasty number
   (i) $4x(a,a)$ & $y(a,a)$
   (ii) $2z(a,a)$ & $x(a,a)$
   (iii) $w(a,a)$

Pattern – 2

Instead of (5) write 26 as

$$26 = (5+i)(5-i) \quad (7)$$

Following a similar procedure as in pattern – 2, the solutions for (3) are as follows,

$$u = u(a,b) = 5a^2 - 5b^2 - 2ab$$

$$v = v(a,b) = a^2 - b^2 + 10ab \quad (8)$$

In view of (2) and (8) the solutions of (1) are obtained as

$$x = x(a,b) = 12a^2 - 12b^2 + 16ab$$

$$y = y(a,b) = 8a^2 - 8b^2 - 24ab$$

$$z = z(a,b) = 24a^2 - 24b^2 + 32ab$$

$$w = w(a,b) = 16a^2 - 16b^2 - 48ab$$

$$T = a^2 + b^2$$

Properties

1. $12T(b^2, b^2-1) + x(b^2, b^2-1) - w(b^2, b^2-1) - 768FN_b^4$
$-288(4DF_b) + 2T_{4,2b} - CP_3^6 + J_5 = 0$
2. $x(2a+1), 2a) - 3y(2a+1), 2a) + z(2a+1), 2a) - 480Pr_a \equiv 24 \pmod{192}$
3. $w(a(a+1), (a+2)(a+3)) + 1152\, Pt_a + 192\, OH_a + T_{4,24a} \equiv 576 \pmod{896}$
4. $3w(b(b+1), b+2) - 2z(b(b+1), b+2) + 1248Tet_b = 0$
5. $8T(a^2, a^2)$ is a bi-quadratic number
6. $y(b, 3b-1) + T_{4,8b} - 8GnO_b + 48T_{5,a} \equiv 0 \pmod{32}$
7. $6x(7b^3 - 4b, 1) - 9y(7b^3 - 4b, 1) - 936CP_b^{14} = 0$
8. $y(5, b) + z(5, b) - 2T(5, b) + 39Pr_b - S_b - CH_b - 38GnO_b - T_{4,27} + CP_4^6 + Ky_1 = 0$
9. $w(3,3) + T(3,3) + J_{10} + j_6 + Ky_1 \equiv 0 \pmod{1}$
10. Each of the following expressions represents a nasty number

(i) $y(a, a)$

(ii) $2z(a, a)$ & $w(a, a)$

(iii) $4x(a, a)$

Pattern – 4

Rewrite (3) as $26T^2 - v^2 = u^2 * 1$ (9)

Assume $u = 26a^2 - b^2$ (10)

Write (1) as = $(\sqrt{26} - 5)\,(\sqrt{26} + 5)$ (11)

Using (10) and (11) in (9) and employing the method of factorization, we write

$$\sqrt{26}\,T + v = (\sqrt{26} - 5) \qquad (\sqrt{26}\,a + b)^2$$

Equating the rational and irrational parts, we have

$$v = v(a,b) = 130a^2 + 5b^2 + 52ab$$
$$T = T(a,b) = 26a^2 + b^2 + 10ab \qquad (12)$$

In view of (2) and (12), the solutions of (1) are obtained as

$$x = x(a,b) = 312a^2 + 8b^2 + 104ab$$
$$y = y(a,b) = -208a^2 - 12b^2 - 104ab$$
$$z = z(a,b) = 624a^2 + 16b^2 + 208ab$$
$$w = w(a,b) = -416a^2 - 24b^2 - 208ab$$
$$T = T(a,b) = 26a^2 + b^2 + 10ab$$

Properties

1. $x(a,11a-9)+y(a,11a-9)+4T(a,11a-9)-80T_{13,a}-T_{418,a}\equiv 0\ (mod\ 207)$
2. $z(b(b+1),b+2)-1248P_B^3-624T_{4,b^2}-1879\ OH_b-10T_{4,8b}\equiv 64\ (mod\ 560)$
3. $w(b,4)+T(b,4)+390Pr_b-CP_7^6+T_{4,5}\equiv 0\ (mod\ 402)$
4. $y(b,5b-3)-w(b,5b-3)-505T_{4,b}-CH_b-208\ T_{7,b}+107GnO_b\equiv 0\ (mod\ 143)$
5. $x(b,2b^2+1)-312\ OH_b-384FN_b^4-15T_{4,5b}\equiv 8\ (mod\ 1)$
6. $x(a^2+a,2a^2-1)+y(a^2+a,2a^2-1)+w(a^2+a,2a^2-1)+840Biq_a-1040CP_a^6-2T_{4,2a}\equiv 28\ (mod\ 208)$
7. $z(2a,a^2)+w(2a,a^2)+2T(2a,a^2)-T_{4,32b^2}-10T_{4,a^2}+CP_a^6=0$
8. $y(a,a)$ is a perfect square
9. $T(2,1)+CP_5^6=0$
10. $y(b(b+1),(b+2)(b+3))-4w(b(b+1),(b+2)(b+3))+T(b(b+1),(b+2)(b+3))-17712Pt_b-18804FN_b^4$ $-7628P_B^5+754T_{4,b}-2550GnO_b+j_9-j_1=0$

Pattern – 5

Introduction of the Linear Transformation

$$v = X+26R,\quad T = X+R,\quad u = 5U$$

In (3) leads to $X = r^2+26S^2$

$$R = 2rS$$

$$U = r^2-26S^2$$

Substituting the above values of X, U and R in (13), the corresponding non-zero distinct integer solutions of (3) are given by

$$v = v(r,s) = r^2+26s^2+52rs$$

$$u = u(r,s) = 5r^2-130s^2$$

$$T = T(r,s) = r^2+26s^2+2rs$$

Thus the corresponding solutions of (1) are found to be

$$x = x(r,s) = 12r^2-208s^2+104rs$$

$$y = y(r,s) = 8r^2-312s^2-104rs$$

$$z = z(r,s) = 24r^2-416s^2+208rs$$

$$w = w(r,s) = 16r^2-624s^2-208rs$$

$$T = T(r,s) = r^2+26s^2+2rs$$

Properties

1. $x(r^2,r+1)+z(r^2,r+1)-T(r^2,r+1)-620P_r^5-420FN_r^4+615\ Obl_r\equiv 650\ (mod\ 685)$
2. $x(r,3r-1)+y(r,3r-1)+z(r,3r-1)+8380T_{4,r}-936GnO_r-416\ T_{5,r}\equiv 0\ (mod\ 3744)$
3. $T(r,(r+1)(r+2))-4P_r^3-12(4DF_r)-56PP_r+26CP_r^6-j_7+Ky_2\equiv 0\ (mod\ 104)$
4. $y(s,5s-3)+208T_{7,s}+7792T_{4,s}+J_{13}+Ky_3-j_0\equiv 0\ (mod\ 9360)$
5. $y(s,(s+1)(s+2))-8T(s,(s+1)(s+2))+720Tet_s+T_{1042,s}\equiv 0\ (mod\ 519)$
6. $w(2r+1,r(r+1))-160\ Obl_r-10T_{4,2}+960SqP_r=0$
7. $z(s,(s+1)(5s-2))-w(s,(s+1)(5s-2))-y(s,(s+1)(5s-2))+3120HP_s-520T_{4,s}=0$
8. $x(7r^2-4,r^2)-312CP_r^{14}-7056(4DF_r)+73T_{4,2r}-3CP_4^6=0$
9. $2x(2s^2-1,s)-z(2s^2-1,s)+T(2s^2-1,s)-2SO_s-T_{4,2s^2}-T_{46,s}-J_1\equiv 0\ (mod\ 21)$
10. $w(1,1)+z(1,1)-CP_{10}^6=0.$

Conclusion

To conclude, one may search for other patterns and their corresponding properties.

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