ORIGINAL RESEARCH PAPER

Mathematics

# CENTERED HEXAGONAL GRACEFUL LABELING OF N-STAR GRAPH

**KEY WORDS:** Centered hexagonal numbers, centered hexagonal graceful labeling, centered hexagonal graceful graph.

**A. Ramalakshmi** Research scholar, Department of Mathematics, The M.D.T. Hindu college, Tirunelveli – 627010, Tamilnadu, India.

**M. P. Syed Ali Nisaya*** Assistant Profesor, Department of Mathematics, The M.D.T. Hindu college, Tirunelveli – 627010, Tamilnadu, India. *Corresponding Author

**ABSTRACT**

Let $G$ be a (p,q) graph. Let V(G)and E(G) denote the vertex set and edge set of G respectively. A centered hexagonal graceful labeling of a graph is an one to one function $f:V(G)\to\{0,1,2,\dots,D_q\}$ that induces a bijection $f^*:E(G)\to\{D_1,D_2,\dots,D_q\}$ of the edges of defined by $f^*(e)=|f(u)-f(v)|,\forall e=uv\in E(G)$ . The graph which admits such a labeling is called a centered hexagonal graceful graph. In this paper, we prove that $n$-star graph is a centered hexagonal graceful graph.

## 1. INTRODUCTION AND DEFINITIONS

The graphs considered in this paper are finite, undirected and without loops or multiple edges. Let $G=(V,E)$ be a graph with p vertices and q edges. Terms not defined here are used in the sense of Harary [2]. For number theoretic terminology [1] is followed.

A graph labeling is an assignment of integers to the vertices or the edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (edges/both) then the labeling is called a vertex (edge/ total) labeling.

There are several types of graph labeling and a detailed survey is found in [3]. In 1967, Rosa [5] presented four hierarchically related labeling of graphs,which he named α,β ,σ and ρ valuations. In 1972 ,β-valuation had been called graceful labeling by Golomb [4]. Ramesh and Syed Ali Nisaya [6] introduced some more polygonal graceful labeling of path .

**Definition 1.1:** The star graph $K_{1,n}$ of order $n+1$ is a tree on $n$ edges with one vertex having degree $n$ and other vertices having degree 1.

**Definition 1.2:** The – star $G$ is the disjoint union of $K_{1,a_1}, K_{1,a_2},\dots,K_{1,a_n}$ where $a_1,a_2,\dots,a_n$ are positive integers and $K_{1,a_i}$ is a star of length $a_i$ for $1\le i\le n$ We denote it by $K_{1,a_1}\cup K_{1,a_2}\cup\dots\cup K_{1,a_n}$ Here $G$ has $a_1+a_2+\dots+a_n+n$ vertices and $a_1+a_2+\dots+a_n$ edges.

**Definition 1.3:** Let $G$ be a $(p,q)$ graph. Let $V(G)$ $E(G)$ denote the vertex set and the edge set of $G$ respectively. A one to one function $f:V(G)\to\{0,1,2,\dots,q\}$ is called a **graceful labeling** of $G$ if the induced edge labeling $f^*:E(G)\to\{1,2,\dots,q\}$ defined by $f^*(e)=|f(u)-f(v)|$, for each edge $e=uv$ of $G$ is also one to one . A graph Gpossessing graceful labeling is called a **graceful graph.**

**Definition 1.4:** A centered hexagonal number is a centered figurate number that represents a hexagon with a dot in the center and all other dots surrounding the center in successive hexagonal layers. The $n^{th}$ centered hexagonal number is found to be $D_n=n^3-(n-1)^3=3n(n-1)+1$ The first few centered hexagonal numbers are 1,7,19,37, 61,91, 127, 169,217,271,331,397 etc.

**Definition 1.5:** A centered hexagonal graceful labeling of a graph $G$ is an one to one function $f:V(G)\to\{0,1,2,\dots,D_q\}$ that induces a bijection $f^*:E(G)\to\{D_1,D_2,\dots,D_q\}$ of the edges of $G$ defined by $f^*(e)=|f(u)-f(v)|,\forall e=uv\in E(G)$ The graph which admits such a labeling is called a centered hexagonal graceful graph.

## 2. MAIN RESULTS

Now, we prove that the $n$-star is a centered hexagonal graceful graph. First, we prove the following three lemmas.

**Lemma 2.1:** The star graph$nK_{1,n}$ is a centered hexagonal graceful graph for all $1\ge n$ .

**Proof:** Note that the graph $K_{1,n}$ has $(n+1)$ vertices and $n$ *edges*. Let $u$ be the unique vertex in one partition of $K_{1,n}$ and $u_1,u_2,\dots,u_n$ be the $n$ vertices in the other.

Define $f:V(K_{1,n})\to\{0,1,2,\dots,D_n\}$ as follows.

$f(u)=0$

$f(u_i)=3i^2-3i+1$ where $1\le i\le n$

Clearly $f$ is one to one and the edge values are $D_1,D_2,\dots,D_n$. Hence $K_{1,n}$ is a centered hexagonal graceful graph.

**Lemma 2.2:** The 2-star graph is a centered hexagonal graceful graph

**Proof:** Let $G=(V,E)$ be a 2-star $K_{1,a_1}\cup K_{1,a_2}$ for all $a_1,a_2\ge 1$ with the vertex set

$V=\{u_{10},u_{20},u_{11},u_{12},\dots,u_{1a_1},u_{21},u_{22},\dots,u_{2a_2}\}$ and the edge set

$E=\{u_{10}u_{1j}:1\le j\le a_1\}\cup\{u_{20}u_{2j}:1\le j\le a_2\}$. Then $G$ has $a_1+a_2+2$ vertices and $a_1+a_2$ edges. Take $a_1+a_2=m$ .

Define $f:V(G)\to\{0,1,2,\dots,D_m\}$ as follows.

$f(u_{i0})=i-1$, where $i=1,2$

$f(u_{1j})=3(m^2-2mj+j^2+m-j)+1$, where $1\le j\le a_1$

$f(u_{2j})=3(m^2-2a_1m-2jm+a_1^2+2a_1j+j^2+m-a_1-j)+2$, where $1\le j\le a_2$

We shall prove that $G$ admits centered hexagonal graceful labeling. From the definition, it is clear that $\max_{v\in V(G)} f(v)$ is $D_m$ and also $f(v)\in\{0,1,2,\dots,D_m\}$.

Also from the definition, all the vertices of $G$ have different labeling. Hence $f$ is one to one.

It remains to show that the edge values are of the form $\{D_1,D_2,\dots,D_m\}$ .

The induced edge function $f^*:E(G)\to\{D_1,D_2,\dots,D_m\}$ is defined as follows.

$f^*(u_{10}u_{1j})=3(m^2-2mj+j^2+m-j)+1$, where $1\le j\le a_1$

$f^*(u_{20}u_{2j})=3(m^2-2a_1m-2jm+a_1^2+2a_1j+j^2+m-a_1-j)+1$ , where $1\le j\le a_2$

Clearly $f^*$ is a bijection and $f^*(E(G))=\{D_1,D_2,\dots,D_m\}$. Therefore $G$ admits centered hexagonal graceful labeling. Hence the graph 2-star $K_{1,a_1}\cup K_{1,a_2}$ for all $a_1,a_2\ge 1$ is a centered hexagonal graceful graph.

**Lemma 2.3:** The 3-star graph is a centered hexagonal graceful graph.

**Proof:** Let $G=(V,E)$ be a 3-star $K_{1,a_1}\cup K_{1,a_2}\cup K_{1,a_3}$ for all $a_1,a_2,a_3\ge 1$ with the vertex set and the edge set respectively.

$V=\{u_{10},u_{11},u_{12},\dots,u_{1a_1},u_{20},u_{21},u_{22},\dots,u_{2a_2},u_{30},u_{31},u_{32},\dots,u_{3a_3}\}$ and

$E=\{u_{10}u_{1j}:1\le j\le a_1\}\cup\{u_{20}u_{2j}:1\le j\le a_2\}\cup\{u_{30}u_{3j}:1\le j\le a_3\}$. Then $G$ has $a_1+a_2+a_3+3$ vertices and $a_1+a_2+a_3$ edges. Take $a_1+a_2+a_3=m$ .

Define $f:V(G)\to\{0,1,2,\dots,D_m\}$ as follows.

$f(u_{i0})=i-1$, where $i=1,2,3$

$f(u_{1j}) = 3(m^2 - 2mj + j^2 + m - j) + 1$ , where $1 \le j \le a_1$

$f(u_{2j}) = 3(m^2 - 2a_1 m - 2jm + a_1^2 + 2a_1 j + j^2 + m - a_1 - j) + 2$ , where $1 \le j \le a_2$

$f(u_{3j}) = 3(m^2 - 2m(a_1 + a_2) - 2jm + a_1^2 + a_2^2 + 2a_1 a_2 + 2j(a_1 + a_2) + j^2 + m - (a_1 + a_2) - j) + 3$ , where $1 \le j \le a_3$

We shall prove that $G$ admits centered hexagonal graceful labeling. From the definition, it is clear that $\max_{v \in V(G)} f(v)$ is $D_m$ and also $f(v) \in \{0, 1, 2, \ldots, D_m\}$.

Also from the definition, all the vertices of $G$ have different labeling. Hence $f$ is one to one.

It remains to show that the edge values are of the form $\{D_1, D_2, \ldots, D_m\}$ . The induced edge function $f^*: E(G) \to \{D_1, D_2, \ldots, D_m\}$ is defined as follows.

$f^*(u_{10}u_{1j}) = 3(m^2 - 2mj + j^2 + m - j) + 1$, where $1 \le j \le a_1$

$f^*(u_{20}u_{2j}) = 3(m^2 - 2a_1 m - 2jm + a_1^2 + 2a_1 j + j^2 + m - a_1 - j) + 1$ , where $1 \le j \le a_2$

$f^*(u_{30}u_{3j}) = 3(m^2 - 2m(a_1 + a_2) - 2jm + a_1^2 + a_2^2 + 2a_1 a_2 + 2j(a_1 + a_2) + j^2 + m - (a_1 + a_2) - j) + 2$,

where $1 \le j \le a_3$

Clearly $f^*$ is a bijection and $f^*(E(G)) = \{D_1, D_2, \ldots, D_m\}$. Therefore $G$ admits centered hexagonal graceful labeling. Hence the graph 3-star $K_{1,a_1} \cup K_{1,a_2} \cup K_{1,a_3}$ for all $a_1, a_2, a_3 \ge 1$ is a centered hexagonal graceful graph.

**Theorem 2.4:** The $n$- star graph is a centered hexagonal graceful graph.

**Proof:** Let $G = (V, E)$ be a $n$-star $K_{1,a_1} \cup K_{1,a_2} \cup \ldots \cup K_{1,a_n}$ for all $a_1, a_2, \ldots, a_n \ge 1$ with

the vertex set and the edge set respectively.

$V = \{u_{10}, u_{11}, u_{12}, \ldots, u_{1a_1}, u_{20}, u_{21}, u_{22}, \ldots, u_{2a_2}, \ldots, u_{n0}, u_{n1}, u_{n2}, \ldots, u_{na_n}\}$ and

$E = \{u_{10}u_{1j} : 1 \le j \le a_1\} \cup \{u_{20}u_{2j} : 1 \le j \le a_2\} \cup \ldots \cup \{u_{n0}u_{nj} : 1 \le j \le a_n\}$. Then $G$ has

$a_1 + a_2 + \ldots + a_n + n$ vertices and $a_1 + a_2 + \ldots + a_n$ edges. Take $a_1 + a_2 + \ldots + a_n = m$ .

Define $f: V(G) \to \{0, 1, 2, \ldots, D_m\}$ as follows.

$f(u_{i0}) = i - 1$, where $1 \le i \le n$

$f(u_{1j}) = 3(m^2 - 2mj + j^2 + m - j) + 1$ , where $1 \le j \le a_1$

$f(u_{2j}) = 3(m^2 - 2a_1 m - 2jm + a_1^2 + 2a_1 j + j^2 + m - a_1 - j) + 2$ , where $1 \le j \le a_2$

$f(u_{3j}) = 3(m^2 - 2m(a_1 + a_2) - 2jm + a_1^2 + a_2^2 + 2a_1 a_2 + 2j(a_1 + a_2) + j^2 + m - (a_1 + a_2) - j) + 3$ , where $1 \le j \le a_3$ and so on.

$$f(u_{nj}) = 3\begin{pmatrix} m^2 - 2m(a_1 + a_2 + \ldots + a_{n-1}) - 2jm + (a_1^2 + a_2^2 + a_{n-1}^2) \\ + 2j(a_1 + a_2 + \ldots + a_{n-1}) + j^2 + m - (a_1 + a_2 + \ldots + a_{n-1}) \\ - j + 2[(a_1 a_2 + a_1 a_3 + \ldots + a_1 a_{n-1}) + (a_2 a_3 + \ldots + a_2 a_{n-1})] \end{pmatrix} + n$$

We shall prove that $G$ admits centered hexagonal graceful labeling. From the definition, it is clear that $\max_{v \in V(G)} f(v)$ is $D_m$ and also $f(v) \in \{0, 1, 2, \ldots, D_m\}$.

Also from the definition, all the vertices of $G$ have different labeling. Hence $f$ is one to one.

It remains to show that the edge values are of the form $\{D_1, D_2, \ldots, D_m\}$ .

The induced edge function $f^*: E(G) \to \{D_1, D_2, \ldots, D_m\}$ is defined as follows.

$f^*(u_{10}u_{1j}) = 3(m^2 - 2mj + j^2 + m - j) + 1$ , where $1 \le j \le a_1$

$f^*(u_{20}u_{2j}) = 3(m^2 - 2a_1 m - 2jm + a_1^2 + 2a_1 j + j^2 + m - a_1 - j) + 1$ , where $1 \le j \le a_2$

$f^*(u_{30}u_{3j}) = 3(m^2 - 2m(a_1 + a_2) - 2jm + a_1^2 + a_2^2 + 2a_1 a_2 + 2j(a_1 + a_2) + j^2 + m - (a_1 + a_2) - j) + 3$

, where $1 \le j \le a_3$ and so on.

$$f^*(u_{n0}u_{nj}) = 3\begin{pmatrix} m^2 - 2m(a_1 + a_2 + \ldots + a_{n-1}) - 2jm + (a_1^2 + a_2^2 + a_{n-1}^2) \\ + 2j(a_1 + a_2 + \ldots + a_{n-1}) + j^2 + m - (a_1 + a_2 + \ldots + a_{n-1}) - j + \\ 2[(a_1 a_2 + a_1 a_3 + \ldots + a_1 a_{n-1}) + (a_2 a_3 + \ldots + a_2 a_{n-1})] \end{pmatrix} + (n - 1)$$

Clearly $f^*$ is a bijection and $f^*(E(G)) = \{D_1, D_2, \ldots, D_m\}$. Therefore $G$ admits centered hexagonal graceful labeling. Hence the graph $n$-star $K_{1,a_1} \cup K_{1,a_2} \cup \ldots \cup K_{1,a_n}$ for all $a_1, a_2, \ldots, a_n \ge 1$ is a centered hexagonal graceful graph.

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