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ORIGINAL RESEARCH PAPER

Mathematics

FAILURE OF PARABOLIC GROWTH FUNCTIONS AT HIGH GROWTH RATES.

KEY WORDS:

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ABSTRACT

Growth functions are mathematical functions that are used to predict the population growth of any initially given sample set. Growth functions are used in an extensive range of fields, including mapping population growth of a demographic or for laboratory population samples, artificial neural networks, prediction of sale of a staple product and many more. The most popular growth function is given by the Verhulst model.

I. GROWTH FUNCTIONS.

Logistic Growth Model or Velhurst growth model is mathematically given by:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \qquad (1)$$

Where P is the population as a function of time, r is the growth rate, and K is the carrying capacity which restricts the function from blowing up exponentially. Before the above equation is farther dismantled, first we must observe how we arrive at that equation in the first place.

Suppose we have to calculate the population growth of a sample with a known initial population. If we take the Population 'P' as a function of time and assume 'r' is the number at which it will multiply at the next time step. Then this growth can be intuitively modeled as:

$$\frac{dP}{dt} = rP \qquad (2)$$

which shows that the rate at which the population grows over time depends on the population itself and at the factor at which it is multiplying. Solution of the above equation is

$P = P_0 e^{rt}$ where P_0 is the initially known population.

This is an extremely ideal case, where population increases exponentially and no external restrictions are considered. Therefore to take into account the environmental or any other restriction equation (2) can be rewritten as

$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$ where K is carrying capacity, which is a number that readily decreases the growth rate when the population approaches the K.

$$P = \frac{K}{1 + A\exp(-rt)} \qquad (3)$$

The constant A can be determined by defining $P(t=0) = P_0$, therefore

$$A = \frac{K - P_0}{P_0}$$

This shows that the population will continue to grow exponentially until it approaches K, after that it will proceed asymptotically with K.

Now, we are going to plot the equilibrium value i.e. the value at which the growth stabilizes after a sufficient increment in time with a known initial population with respect to time.

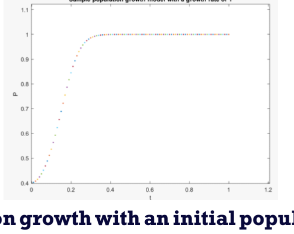

***Fig 1*: Population growth with an initial population of 0.4**

For simplicity, we assume 1=K i.e. the population is normalized within 0 to 1.

Therefore for large increments in time equation (1) can be rewritten as:

$$\frac{\Delta P}{\Delta t} = rP(1-P)$$

Now if we take each time increment as unity i.e. $\Delta t = 1$, and P_0 as the initial population then,

$$\Delta P = rP(1-P), \qquad (4)$$

Here ΔP is the increment in population after each unit time step. Since it is seen that the growth is exponential before the population approaches the carrying capacity, the increment can be written as,

$\Delta P = P_n - P_{n-1} \approx P_n$; where n denotes the n[th] time step. Therefore eq. (3) can be written as,

$$P_n = rP_{n-1}(1 - P_{n-1}) \qquad (5)$$

Eq. 5 gives the same result as eq. 3 as long as the assumptions $K = 1$ and $\Delta t = 1$ holds. Thus it can be inferred that eq. 5 is an iterative corollary of eq. 3.

II) DEVIATION FROM EXPECTED BEHAVIOR AT HIGH GROWTH RATES.

Growth rate r, defines how much the population will increase in the next iteration. Mathematically r could vary from 1 i.e. no increase in population to any value only limited by the sample's ability to multiply. In the next graph, fig 2. the equilibrium value of the population after a large iteration for a specific growth rate, r is plotted against the growth rate itself.

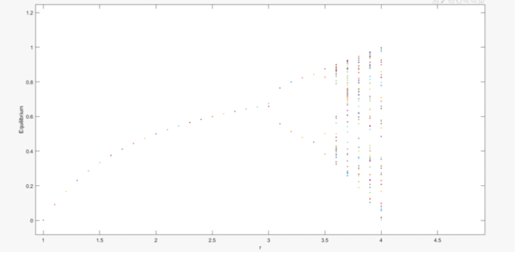

Fig 2 Equilibrium population vs growth rate (r) up to r=4.

This graph is extremely informative. The first part of the graph proceeds as expected as the growth rate increases so does the equilibrium value. At r=3, the equilibrium point bifurcates i.e. the equilibrium population oscillates between two values. At around r=3.5 the points bifurcates again i.e. the equilibrium value oscillates between four values. These bifurcations have been observed to high accuracy in nature. The bifurcation keeps on happening until r reaches a value of r=3.7. After this value, complete chaos takes over and the periodicity vanishes.

This chaotic nature progresses with increasing r, with brief windows of periodicity. But at r around 4.5, the equilibrium value breaks the normalization, which restricts the value to 1. This means after a certain passage of time, the population goes over one, which was restricted by the Carrying Capacity K.

When the population goes over 1, it can be seen from eq 5. that the predicted population will quickly drop to - infinity. And since eq 5. is just a corollary of eq. 3 the same interpretation can be made for eq 3. It can be directly seen in eq. 3

$P = \frac{K}{1 + A\exp(-rt)}$, for small values of r, P will be restricted by K.

But if r is a very large value, then P will no longer be controlled by K and it will be larger than the carrying capacity. Which will defeat the purpose of introducing K in the first place. This problem arises due to the fact that that the carrying capacity K in itself isn't enough to normalize the population and it is a static value. This static nature of K disables us to change the normalization parameter of the function.

Therefore, a new method of normalizing P is required which takes into account large values of r. This can be done by modifying the definition of K. Where K is some function of r, this introduces a dynamic nature in K which will allow it to re-normalize the function when it approaches a high value of r.