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Mathematics

GENERALIZED SEMI STAR b-SEPARATION AXIOMS IN TOPOLOGICAL SPACES

KEY WORDS: gs*b-T_k(k=0,1,2,1/2), gs*b-D_k(k=0,1,2).

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ABSTRACT

The aim of this paper is to introduce the new class of spaces via gs*b-open sets and gs*b-difference sets. Also, we study some basic properties and their characterizations.

1. INTRODUCTION

D.Andrijevic[1] introduced the concept of b-open sets and characterized its topological properties. Caldas and Jafari[2] introduced and studied b-T_0,b-T_1,b-T_2,b-D_0,b-D_1 and b-D_2 via b-Open sets after that Keskin and Noiri[3] introduced the notion of b-$T_{1/2}$. In 1996, Andrijevic gave a new type of generalized closed sets in topological spaces called b-closed sets and we extend this concept into gs*b-open sets.

In this paper, we introduce a new classes of spaces called gs*b-T_k spaces, for k=0,1,2,1/2,gs*b-D_k spaces, for k=0,1,2 and gs*b-spaces. Also, we study some basic properties and their various characterizations.

2. PRELIMINARIES

Throughout this paper (X,τ) represents a topological space on which no separation axiom is assumed unless otherwise mentioned. (X,τ) will be replaced by X if there is no changes of confusion. For a subset A of a topological space X, cl(A) and int(A) denote the closure of A and the interior of A respec tively. We recall the following definitions and results.

DEFINITION 2.1[1]:
Let (X,τ) be a topological space. A subset A of the space X is said to be b-open if A int(cl(A)) cl(int(A)) and b-closed if int(cl(A)) cl(int(A)) A.

DEFINITION 2.2[6]:
Let (X,τ) be a topological space. A subset Aof X is said to be generalized closed (briefly g-closed) if cl(A) ⊆ U whenever A⊆U and U is open in (X,τ). The complement of g-closed set is g-open.

DEFINITION 2.3[4]: If A is a subset of X,
(i) The generalized closure of A is defined as the intersection of all g-closed sets in X containing A and is denoted by cl*(A).
(ii) The generalized interior of A is defined as the union of all g-open sets in X that are contained in A and is denoted by int*(A).

DEFINITION 2.4:
Let (X,τ) be a topological space. A subset A of the space X is said to be semi*-open[5] if A⊆cl*(int(A)) and semi*-closed [4] if int*(cl(A))⊆A.

DEFINITION 2.5[4]:
Let (X,τ) be a topological space and A⊆X. Then
(i) The semi*-closure of A is defined as the intersection of all semi*-closed sets in X containing A. It is denoted by s*cl(A).
(ii) The semi*-interior of A is defined as the union of all semi*-open sets contained in A. It is denoted by s*int(A).

DEFINITION 2.6[7]:
A subset A of a topological space (X,τ) is said to be generalized semi star b-closed (briefly gs*b-closed) if s*cl(A)⊆U whenever A⊆U and U is b-open in (X,τ).

DEFINITION 2.7[7]:
A subset A of (X,τ) is said to be generalized semi star b-open (briefly gs*b-open) set if its complement X\A is gs*b-closed in X. The family of all gs*b-open sets in X is denoted by gs*b-O(X).

DEFINITION 2.8[8]:
Let A be a subset of a topological space (X,τ). Then the union of all gs*b-open sets contained in A is called the gs*b-interior of A and it is denoted by gs*bintA. That is, gs*bint(A)= {V:V ⊆A and V∈gs*b-O(X)}.

DEFINITION 2.9[8]:
Let A be a subset of a topological space (X,τ). Then the intersection of all gs*b-closed sets in X containing A is called the gs*b-closure of A and it is denoted by gs*bcl (A). That is, gs*bcl (A)= ∩F:A⊆{F and F ∈gs*b-C(X)}.

THEOREM 2.10[8]:
Let A be a subset of a topological space (X,τ). Then
1) A is gs*b-open if and only if gs*bint(A)=A
2) A is gs*b-closed if and only if gs*bcl(A)=A.

THEOREM 2.11[7]:
For every element x in a space X, X–{x} is gs*b-closed or b-open.

DEFINITION 2.12[8]:
Let X be a topological space and let x∈X. A subset N of X is said to be a gs*b-neighborhood (shortly, gs*b-nbhd) of x if there exists a gs*b-open set U such that x∈U⊆N.

THEOREM 2.13[1]: Every closed set is b-closed.

3. GENERALIZED SEMI STAR b-T_K SPACES

DEFINITION 3.1:
A topological space (X,τ) is said to be
(I) gs*b-T_0 if for each pair of distinct points x, y in X, there exist a gs*b-open set U in X such that either x∈U and y∉U or x∉U and y∈U.
(ii) gs*b-T_1 if for each pair of distinct points x, y in X, there exist two gs*b-open sets U and V in X such that x∈U but

$y \notin U$ and $x \notin V$ but $y \in V$.

(iii) gs*b-T_2 if for each pair of distinct points x, y in X, there exist two disjoint gs*b-open sets U and V in X such that $x \in U$ but $y \notin U$ and $x \notin V$ but $y \in V$.

(iv) gs*b-$T_{1/2}$ if every gs*b-closed set is b-closed.

(v) gs*b-space if every gs*b-open set is open.

THEOREM 3.2:
A topological space (X,τ) is gs*b-T_0 if and only if for each pair of distinct points x, y in X, $gs^*bcl(\{x\}) \neq gs^*bcl(\{y\})$.

PROOF: NECESSITY:
Suppose X is gs*b-T_0 and x,y are any two distinct points of X. Then there exists a gs*b-open set U containing x or y, say x but not y. Since U is gs*b-open, $X\backslash U$ is a gs*b-closed set which does not contain x but contains y. Since $gs^*bcl(\{y\})$ is the smallest gs*b-closed set containing y, $gs^*bcl(\{y\}) \subseteq X\backslash U$. Then $x \notin gs^*bcl(\{y\})$. Hence $gs^*bcl(\{x\}) \neq gs^*bcl(\{y\})$.

SUFFICIENCY:
Suppose that $x,y \in X$ with $x \neq y$ and $gs^*bcl(\{x\}) \neq gs^*bcl(\{y\})$. Then there exists a point $z \in X$ such that $z \in gs^*bcl(\{x\})$ but $z \notin gs^*bcl(\{y\})$. Now, we claim that $x \notin gs^*bcl(\{y\})$. If $x \in gs^*bcl(\{y\})$, then $gs^*bcl(\{x\}) \subseteq gs^*bcl(\{y\})$. This implies, $z \in gs^*bcl(\{y\})$, which contradicts $z \notin gs^*bcl(\{y\})$. Therefore $x \notin gs^*bcl(\{y\})$. Since $gs^*bcl(\{y\})$ is gs*b-closed set containing y but not x, then $X\backslash gs^*bcl(\{y\})$ is a gs*b-open set containing x but not y. Hence X is a gs*b-T_0 space.

THEOREM 3.3:
A topological space (X,τ) is gs*b-T_1 if and only if the singletons are gs*b-closed sets.

PROOF:
Let (X,τ) be a gs*b-T_1 space and x be any point of X. Let $y \in X\backslash\{x\}$. Then $x \neq y$ and so there exists a gs*b-open set U_y containing y but not x. That is $y \in U_y \subseteq X\backslash\{x\}$. This implies, $X\backslash\{x\} = \cup\{U_y / y \in X\backslash\{x\}\}$. Since the union of gs*b-open sets is gs*b-open, then $X\backslash\{x\}$ is gs*b-open containing y but not x. Hence $\{x\}$ is gs*b-closed in X. Conversely, suppose $\{p\}$ is gs*b-closed, for every $p \in X$. Let $x,y \in X$ with $x \neq y$. Then $y \in X\backslash\{x\}$ and $x \in X\backslash\{y\}$. Since $\{x\}$ and $\{y\}$ are gs*b-closed sets in X, then $X\backslash\{x\}$ and $X\backslash\{y\}$ are gs*b-open sets in X. Thus, we have a gs*b-open set containing x but not y and a gs*b-open set containing y but not x. Hence X is a gs*b-T_1 space.

THEOREM 3.4:
A topological space (X,τ) is gs*b-$T_{1/2}$ if each singleton $\{x\}$ of X is either b-closed or b-open.

PROOF: Let (X,τ) is a gs*b-$T_{1/2}$ space.

CASE (I):
Suppose $\{x\}$ is not b-closed. Then $X\backslash\{x\}$ is not b-open. By Theorem 2.11, $X\backslash\{x\}$ is gs*b-closed. Since X is a gs*b-$T_{1/2}$ space, then $X\backslash\{x\}$ is b-closed and hence $\{x\}$ is b-open.

CASE (II):
Suppose $\{x\}$ is not b-open. Then $X\backslash\{x\}$ is not b-closed. By Theorem 2.11, $X\backslash\{x\}$ is gs*b-open in X. Since X is a gs*b-$T_{1/2}$ space, then $X\backslash\{x\}$ is b-open and hence $\{x\}$ is b-closed.

THEOREM 3.5:
The following statements are equivalent for a topological space X.

(i) X is gs*b-T_2.

(ii) For each $x \in X$ and $y \neq x$, there exists a gs*b-open set U containing x such that $y \notin gs^*bcl(U)$.

(iii) For each $x \in X$, $\cap\{gs^*bcl(U) / U \in gs^*b\text{-}O(X,\tau)$ and $x \in U\} = \{x\}$.

PROOF:
(i)⇒(ii): Suppose X is gs*b-T_2. Then for $x,y \in X$ with $x \neq y$. Then there exists disjoint gs*b-open sets U and V containing x and y respectively. Since V is gs*b-open, then $X\backslash V$ is gs*b-closed containing U. Hence $gs^*bcl(U) \subseteq X\backslash V$. Since $y \in V$, then $y \notin X\backslash V$ and hence $y \notin gs^*bcl(U)$.

(ii)⇒(iii): If there exists an element $y \neq x$ in X such that $y \in \cap\{gs^*bcl(U) / U \in gs^*b\text{-}O(X)$ and $x \in U\}$, then $y \in gs^*bcl(U)$ for every gs*b-open set U containing x. This contradicts our assumption. So there exists no such an element y. This proves (iii).

(iii)⇒(I): Let $x,y \in X$ with $x \neq y$. Then by our assumption, there exists a gs*b-open set U containing x such that $y \notin gs^*bcl(U)$. Let $V = X\backslash gs^*bcl(U)$. Then V is gs*b-open set containing y. Also $x \in U$ and $U \cap V = \phi$. Thus we have a disjoint gs*b-open sets U and V containing x and y respectively. Hence X is a gs*b-T_2 space.

REMARK 3.6: Every gs*b-T_2 space is gs*b-T_1.

THEOREM 3.7: Every gs*b-space is gs*b-$T_{1/2}$.

PROOF:
Let (X,τ) be a gs*b-space and A be any gs*b-closed set in X. Then $X\backslash A$ is gs*b-open in X. Since X is gs*b-space, then $X\backslash A$ is open in X and so A is closed. By Theorem 2.13, A is b-closed. This shows that X is gs*b-$T_{1/2}$.

4. GENERALIZED SEMI STAR b-D_K SPACES

DEFINITION 4.1:
A subset A of a topological space X is called a gs*b-difference set(briefly gs*b-D-set) if there exists $U, V \in gs^*b\text{-}O(X)$ such that $U \neq X$ and $A = U\backslash V$.

THEOREM 4.2: Every proper gs*b-open set is gs*b-D-set.

PROOF:
Let A be any proper gs*b-open subset of a topological space X. Take $U = A$ and $V = \varphi$. Then $A = U\backslash V$ and $U \neq X$. Hence A is gs*b-D-set.

REMARK 4.3:
The converse of the above theorem need not be true which is shown in the following example.

EXAMPLE 4.4:
Let $X = \{a,b,c,d\}$ with a topology $\tau = \{\varphi, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X\}$. Then $gs^*b\text{-}O(X,\tau) = \{\varphi, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, X\}$. Take $U = \{a,b,d\}$ and $V = \{a,b,c\}$. Then $U \neq X$ and $A = U\backslash V = \{a,b,d\}\backslash\{a,b,c\} = \{d\}$ is gs*b-D-set but not a gs*b-open set.

DEFINITION 4.5: A topological space (X,τ) is said to be

(I) gs*b-D_0 if for any pair of distinct points x and y of X there exist a gs*b-D-set of X containing x but not y or a gs*b-D-set of X containing y but not x.

(ii) gs*b-D_1 if for any pair of distinct points x and y of X there exist a gs*b-D-set of X containing x but not y and a gs*b-D-set of X containing y but not x.

(iii) gs*b-D_2 if for any pair of distinct points x and y of X, there exist disjoint gs*b-D-sets G and E of X containing x and y respectively.

THEOREM 4.6: In a topological space (X,τ),

(i) if (X,τ) is gs*b-T_k, then it is gs*b-D_k, for $k = 0, 1, 2$.

(ii) if (X,τ) is gs*b-D_k, then it is gs*b-D_{k-1}, for $k = 1, 2$.

PROOF:

(i) First we prove the result for $k=0$. Suppose (X,τ) is gs*b-T_0. Then for each pair of distinct points x, y in X, there exists a gs*b-open set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. By Theorem 4.2, U is gs*b-D-set in X. Then we have for each pair of distinct points x, y in X, there exists a gs*b-D-set U such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. Hence (X,τ) is a gs*b-D_0 space. Similarly we can prove that every gs*b-T_k space is gs*b-D_k space, for k=1,2.

(ii) Let k=2. Suppose (X,τ) is a gs*b-D_2 space. Then for any pair of distinct points x and y of X, there exists disjoint gs*b-D-sets U and V of X containing x and y respectively. That is for any pair of distinct points x and y of X, there exists a gs*b-D-set U of X containing x but not y and a gs*b-D-set V of X containing y but not x. Hence (X,τ) is a gs*b-D_1 space. Similarly we can prove that every gs*b-D_1 space is a gs*b-D_0 space.

THEOREM 4.7: A space X is gs*b-D_0 if and only if it is gs*b-T_0.

PROOF: NECESSITY:

Suppose that X is gs*b-D_0. Then for each distinct pair $x, y \in X$ there is a gs*b-D-set G containing x or y, say x but not y. Since G is gs*b-D-set, then there are two gs*b-open sets U_1 and U_2 such that $U_1 \neq X$ and $G=U_1 \backslash U_2$. Since $x \in G$ and $y \notin G$, then $x \in U_1$. For $y \notin G$, we have two cases,

(a) $y \notin U_1$

(b) $y \in U_1$ and $y \in U_2$.

In case(a), $x \in U_1$ and $y \notin U_1$. In case(b), $y \in U_2$ and $x \notin U_2$. Thus in both cases we have for each pair of distinct points x and y in X, there exists a gs*b-open set U_1 containing x but not y or a gs*b-open set U_2 containing y but not x. Hence X is gs*b-T_0.

SUFFICIENCY:

Suppose (X,τ) is gs*b-T_0. Then by Theorem 4.6(i), (X,τ) is gs*b-D_0.

THEOREM 4.8:

A space X is gs*b-D_1 if and only if it is gs*b-D_2.

PROOF: NECESSITY:

Let $x, y \in X$, with $x \neq y$. Then there exist gs*b-D -sets G_1, G_2 in X such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$. Since G_1 and G_2 are gs*b-D-sets, then $G_1 = U_1 \backslash U_2$ and $G_2 = U_3 \backslash U_4$, where U_1, U_2, U_3 and U_4 are gs*b-open sets in X. From $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.

(I) SUPPOSE $X \notin U_3$. FOR $Y \notin G_1$ WE HAVE TWO SUB-CASES:

(a) Suppose $y \notin U_1$. Since $x \in U_1 \backslash U_2$, it follows that $x \in U_1 \backslash (U_2 \cup U_3)$, and since $y \in U_3 \backslash U_4$ we have $y \in U_3 \backslash (U_1 \cup U_4)$. Since the union of gs*b-open sets is gs*b-open set, then $U_2 \cup U_3$ and $U_1 \cup U_4$ are gs*b-open sets. Also $(U_1 \backslash (U_2 \cup U_3)) \cap (U_3 \backslash (U_1 \cup U_4)) = \varphi$. Thus we have disjoint gs*b-D-sets $U_1 \backslash (U_2 \cup U_3)$ and $U_3 \backslash (U_1 \cup U_4)$ containing x and y respectively.

(b) If $y \in U_1$ and $y \in U_2$, we have $x \in U_1 \backslash U_2$, and $y \in U_2$. Also $(U_1 \backslash U_2) \cap U_2 = \varphi$. Thus we have disjoint gs*b-D-sets $U_1 \backslash U_2$ and U_2 containing x and y respectively.

(ii) Suppose $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \backslash U_4$ and $x \in U_4$. Hence $(U_3 \backslash U_4) \cap U_4 = \varphi$. Thus we have disjoint gs*b-D-sets U_4 and $U_3 \backslash U_4$ containing x and y respectively. Hence X is gs*b-D_2.

SUFFICIENCY:

Suppose X is gs*b-D_2. Then by Theorem 4.6(ii), X is gs*b-D_1.

DEFINITION 4.9:

A point $x \in X$ which has only X as the gs*b-neighborhood is called a gs*b-neat point.

THEOREM 4.10:

For a gs*b-T_0 space (X,τ) the following are equivalent:

(I) (X,τ) is gs*b-D_1.

(ii) (X,τ) has no gs*b-neat point.

PROOF:

(i)⇒(ii). Since (X,τ) is gs*b-D_1, then each point x of X is contained in a gs*b-D-set $A = U \backslash V$ and thus in U. By definition $U \neq X$. This implies that x is not a gs*b-neat point.

(ii)⇒(I) Suppose (X,τ) has no gs*b-neat point. Let x and y be distinct points in X. Since X is gs*b-T_0, then there exists a gs*b-open set U containing x or y, say x. Since $y \notin U$, then $U \neq X$. By Theorem 4.2, U is a gs*b-D-set. Since X has no gs*b-neat point, then y is not a gs*b-neat point. This means that there exists a gs*b-neighborhood V of y such that $V \neq X$. Since V is a gs*b-nbhd of y, there exists a gs*b-open set G such that $y \in G \subseteq V$.

Thus $y \in G \backslash U$ but not x. Also $G \backslash U$ is a gs*b-D-set. Hence X is a gs*b-D_1 space.

COROLLARY 4.11:

A gs*b-T_0 space X is not gs*b-D_1 if and only if there is a unique gs*b-neat point in X.

PROOF:

Suppose (X, τ) be a gs*b-T_0 space. But (X,τ) is not a gs*b-D_1. Then by the above theorem (X,τ) has a gs*b-neat point. Now we have to prove the uniqueness. Suppose x and y are two different gs*b-neat points in X. Since X is gs*b-T_0, at least one of x and y, say x, has a gs*b-open set U containing x but not y. Then U is a gs*b-nbhd of x and $U \neq X$. Therefore x is not a gs*b-neat point which contradicts x is a gs*b-neat point. Hence x=y.

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