ORIGINAL RESEARCH PAPER

Mathematics

CALCULATION OF ELLIPSE CIRCUMFERENCE AS A CYLINDRICAL SECTION

KEY WORDS: Ellipse circumference, Cylindrical sections

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ABSTRACT

In this article, ellipse is studied as cylindrical section and a simple formula is presented to calculate the circumference of an ellipse approximately.

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Introduction

Although there is a simple and exact formula for circumference of a circle with radius r, $2\pi r$, exact formula for circumference of an ellipse is complex. Let semi-major and semi-minor axes of an ellipse are a and b respectively and $x=a\cos(\Phi)$ and $y=b\sin(\Phi)$ are its parametric equations, $0\le\Phi\le\frac{\pi}{2}$, then the perimeter L of the ellipse is equal to:

$$L = \int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2(\Phi) + b^2\cos^2(\Phi)}\, d\Phi \quad [1]$$

Several approximations have been presented to calculate the ellipse circumference. Some of them are pointed below:

- Kepler (1609), $L = 2\pi\sqrt{ab}$
- Euler (1773), $L = \pi\sqrt{2(a^2 + b^2)}$
- Sipos (1792), $L = 2\pi\frac{(a+b)^2}{(\sqrt{a}+\sqrt{b})^2}$
- Muir (1883), $L = 2\pi(\frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{2})^{\frac{2}{3}}$
- Peano (1889), $L = \pi(\frac{3(a+b)}{2} - \sqrt{ab})$
- Lindner (1904), $L = \pi(a + b)(1 + \frac{h}{8})^2$, where $h = (\frac{a-b}{a+b})^2$
- Ramanujan II (1914), $L = \pi(a + b)(1 + \frac{3h}{10+\sqrt{4-3h}})$, where $h = (\frac{a-b}{a+b})^2$
- Selmer (1975), $L = \frac{\pi}{4}[(6 + \frac{(a-b)^2}{2(a+b)^2})(a + b) - \sqrt{2(a^2 + 3ab + b^2)}]$
- Almkvist (1978), $L = 2\pi\frac{2(a+b)^2-(\sqrt{a}-\sqrt{b})^4}{(\sqrt{a}+\sqrt{b})^2+2\sqrt{2(a+b)}\sqrt[4]{ab}}$

The newest approximations are:

- Bartolomeu-Michon (2004), $L = \pi\frac{a-b}{a\tan(\frac{a-b}{a+b})}$
- Cantrell II (2004), $L = 4(a + b) - \frac{(8-2\pi)ab}{p(a+b)+(1-2p)\frac{\sqrt{(a+bw)(wa+b)}}{1+w}}$, where $w = 74$ and $p = 0.2410117$
- Sykora-Rivera (2005), $L = 4\frac{\pi ab+(a-b)^2}{a+b}$ [2][3]

I have tried to find a simple approximate formula by analyzing ellipse as the cylindrical section.

Ellipse as a cylindrical section

Figure 1 displays a right cylinder has been intersected by two planes, one of them is vertical to the axis of cylinder (plane A) and the other one is a plane with β angle (plane B).

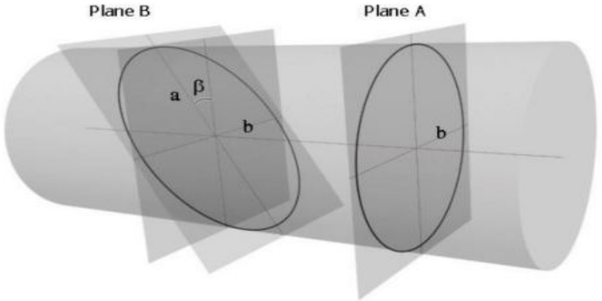

Figure 1: Right regular cylinder intersected by two planes

The curve of intersection on plane A is a circle with radius b and the curve of intersection on plane B is an ellipse with semi-major axis *a* and semi-minor axis *b*. [4]

Regular polygon

Figure 2 displays a regular polygon inscribed in a circle with radius *b*.

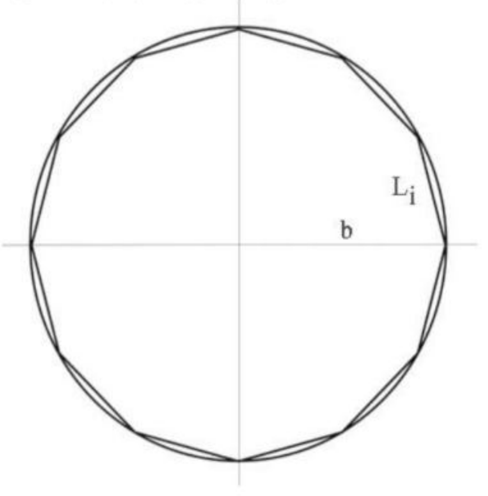

Figure 2: Regular polygon inscribed in a circle

While the sides of the polygon increase, the circumference of the polygon approaches to the circumference of the circle. [5] So, we have:

$Circle\ Circumference = \lim_{n\to\infty} \sum_{i=1}^{n} L_i$, where *n* is the number of polygon sides.

Consider the cylinder of figure 1 with the regular polygon cross section inscribed in a circle (Figure 3).

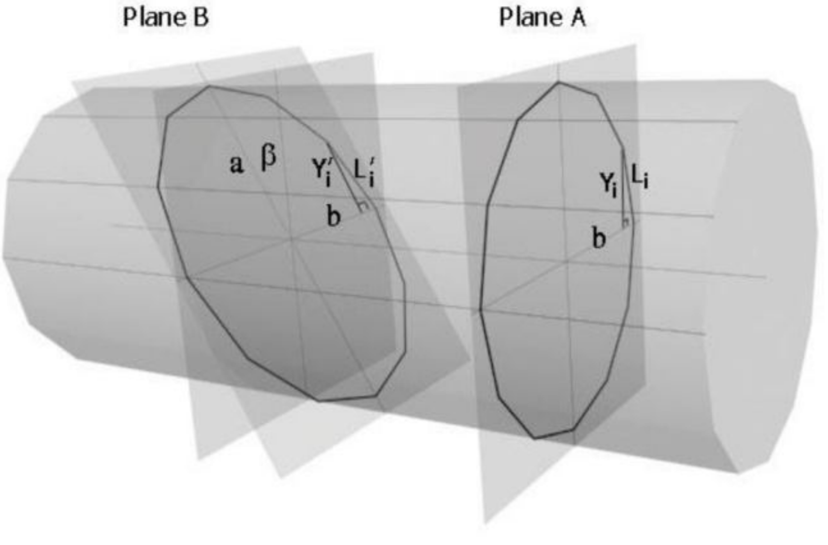

Figure 3: Cylinder with regular polygon cross section

While the sides of the polygon increased, the circumference of section on plane A approaches to circumference of the circle and circumference of the section on plane B approaches to circumference of the ellipse. So, we have:

$Circumference\ of\ Polygon = \sum_{i=1}^{n} L_i'$, where *n* is number of polygon sides.

$$Circumference\ of\ Ellipse = \lim_{n\to\infty} \sum_{i=1}^{n} L_i'$$

For simplicity, I have displayed plane A and B on the same surface (Figure 4) and performed calculations on one quadrant (Figure 5).

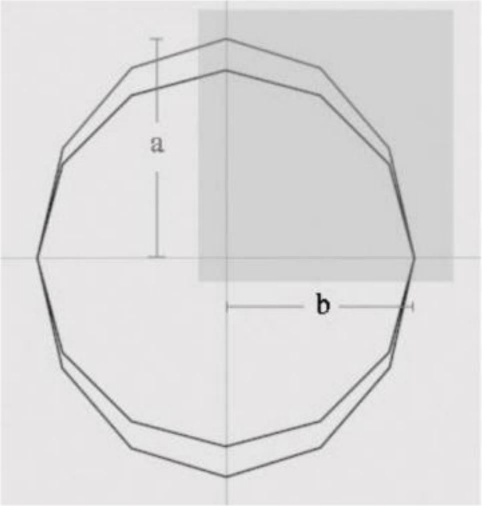

Figure 4: Plane A and B in the same plane

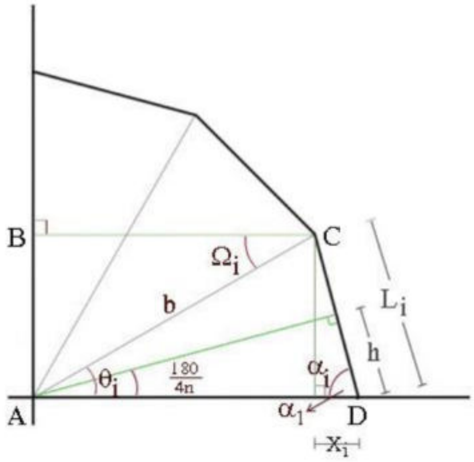

Figure 5: A quadrant of right regular polygon

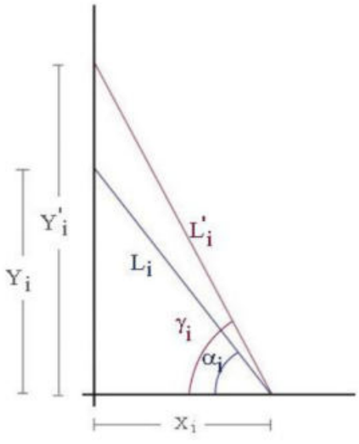

Figure 6: Relationship between L_i and L_i'

We need to calculate L_i' base on L_i. According to figure 3 and figure 5, we have:

$$\cos(\beta) = \frac{b}{a} = \frac{Y_i}{Y_i'} \quad (1)$$

$$\cos(\beta) = \frac{b}{a} = \frac{Y_i}{Y_i'} \rightarrow Y_i' = Y_i.\cos(\beta) \quad (2)$$

$$\sin(\alpha_i) = \frac{Y_i}{L_i} \rightarrow Y_i = L_i.\sin(\alpha_i) \quad (3)$$

$$\sin(\gamma_i) = \frac{Y_i'}{L_i'} \rightarrow Y_i' = L_i'.\sin(\gamma_i) \quad (4)$$

Regarding to 2, 3 and 4, we conclude that:

$$\cos(\beta) = \frac{L_i.\sin(\alpha_i)}{L_i'.\sin(\gamma_i)} \quad (5)$$

$$L_i' = \frac{L_i.\sin(\alpha_i)}{\cos(\beta)\sin(\gamma_i)} \quad (6)$$

$$\cos(\gamma_i) = \frac{X_i}{L_i'} \quad (7)$$

$$\cos(\alpha_i) = \frac{X_i}{L_i} \quad (8)$$

Regarding to 7 and 8, we conclude that:

$$L_i.\cos(\alpha_i) = L_i'.\cos(\gamma_i) \rightarrow \cos(\gamma_i) = \frac{L_i.\cos(\alpha_i)}{L_i'} \quad (9)$$

$$L_i = \frac{L_i'.\cos(\gamma_i)}{\cos(\alpha_i)} \quad (10)$$

$$L_i' = \frac{L_i.\cos(\alpha_i)}{\cos(\gamma_i)} \quad (11)$$

Regarding to 6 and 11, we conclude that:

$$\frac{L_i.\sin(\alpha_i)}{\cos(\beta)\sin(\gamma_i)} = \frac{L_i.\cos(\alpha_i)}{\cos(\gamma_i)} \quad (12)$$

$$\frac{\sin(\alpha_i)}{\cos(\alpha_i)\cos(\beta)} = \frac{L_i.\cos(\alpha_i)}{\cos(\gamma_i)} \quad (13)$$

$$\tan(\gamma_i) = \frac{\tan(\alpha_i)}{\cos(\beta)} \rightarrow tan^2(\gamma_i) = \frac{tan^2(\alpha_i)}{cos^2(\beta)} \quad (14)$$

$$sin^2(\gamma_i) + cos^2(\gamma_i) = 1 \rightarrow \cos(\gamma_i) = \sqrt{1 - sin^2(\gamma_i)} \quad (15)$$

$$\sin^2(\gamma_i) = \frac{\tan^2(\gamma_i)}{1 + \tan^2(\gamma_i)} \quad (16)$$

Regarding to 15 and 16, we conclude that:

$$\cos(\gamma_i) = \sqrt{1 - \frac{tan^2(\gamma_i)}{1 + tan^2(\gamma_i)}} = \sqrt{\frac{1 + tan^2(\gamma_i) - tan^2(\gamma_i)}{1 + tan^2(\gamma_i)}} \quad (17)$$

$$\cos(\gamma_i) = \frac{1}{\sqrt{1 + tan^2(\gamma_i)}} \quad (18)$$

Regarding to 14 and 18, we conclude that:

$$\cos(\gamma_i) = \frac{1}{\sqrt{1 + \frac{tan^2(\alpha_i)}{cos^2(\beta)}}} \quad (19)$$

Regarding to 11 and 19, we conclude that:

$$L_i' = \frac{L_i.\cos(\alpha_i)}{\frac{1}{\sqrt{1 + \frac{tan^2(\alpha_i)}{cos^2(\beta)}}}} \quad (20)$$

$$L_i' = L_i.\cos(\alpha_i)\sqrt{1 + \frac{tan^2(\alpha_i)}{cos^2(\beta)}} \quad (21)$$

$$L_i' = L_i.\cos(\alpha_i)\sqrt{1+\frac{tan^2(\alpha_i)}{cos^2(\beta)}} \quad (21)$$

$$L_i' = L_i\sqrt{cos^2(\alpha_i)+\frac{sin^2(\alpha_i)}{cos^2(\beta)}} \quad (22)$$

From figure 3, we have:

$$Ellipse\ Circumference = 4\times\lim_{n\to\infty}\sum_{i=1}^{n}L_i' \quad (23)$$

Regarding to 22 and 23, we conclude that:

$$Ellipse\ Circumference = 4\times\lim_{n\to\infty}\sum_{i=1}^{n}L_i\sqrt{cos^2(\alpha_i)+\frac{sin^2(\alpha_i)}{cos^2(\beta)}} \quad (24)$$

According to figure 5, in ΔADC we have:

$$2\alpha_1 = 180^{\circ}-\frac{180^{\circ}}{2n} = 180^{\circ}\left(1-\frac{1}{2n}\right) = 180^{\circ}\left(\frac{2n-1}{2n}\right) \quad (25)$$

$$\alpha_1 = \frac{180^{\circ}(2n-1)}{4n} \quad (26)$$

And in ΔABC we have:

$$180^{\circ} = 90^{\circ}+\Omega_i+(n-i)\theta_i = 90^{\circ}+\Omega_i+(n-i)\frac{180^{\circ}}{2n} \quad (27)$$

$$\Omega_i = 180^{\circ}-90^{\circ}-\frac{180^{\circ}(n-i+1)}{2n} = 90^{\circ}-\frac{180^{\circ}(n-i+1)}{2n} \quad (28)$$

$$\Omega_i = \frac{n180^{\circ}-180^{\circ}(n-i+1)}{2n} = \frac{180^{\circ}(n-n+i-1)}{2n} \quad (29)$$

$$\Omega_i = \frac{180^{\circ}(i-1)}{2n} \quad (30)$$

$$\alpha_i = \alpha_1-\Omega_i \quad (31)$$

Regarding to 26, 30 and 31, we conclude that:

$$\alpha_i = \frac{180^{\circ}(2n-1)}{4n}-\frac{180^{\circ}(i-1)}{2n} = \frac{180^{\circ}(2n-1)-360^{\circ}(i-1)}{4n} \quad (32)$$

$$\alpha_i = \frac{180^{\circ}(2n-1-2i+2)}{4n} = \frac{180^{\circ}(2n-2i+1)}{4n} \quad (33)$$

$$\sin\left(\frac{180^{\circ}}{4n}\right) = \frac{h}{b} \to h = b.\sin\left(\frac{180^{\circ}}{4n}\right) \quad (34)$$

$$L_i = 2h \quad (35)$$

Regarding to 34 and 35, we conclude that:

$$L_i = 2b.\sin\left(\frac{180^{\circ}}{4n}\right) \quad (36)$$

Regarding to 24 and 36, we conclude that:

$$Ellipse\ Circumference = 4 \times \lim_{n\to\infty} 2b.\sin(\frac{180^{\circ}}{4n}) \sum_{i=1}^{n} L_i \sqrt{cos^2(\alpha_i) + \frac{sin^2(\alpha_i)}{cos^2(\beta)}} \quad (37)$$

Regarding to 15 and 37, we conclude that:

$$S = \sum_{i=1}^{n} \sqrt{1 - sin^2(\alpha_i) + \frac{sin^2(\alpha_i)}{cos^2(\beta)}} \quad (38)$$

$$S = \sum_{i=1}^{n} \sqrt{1 - \frac{sin^2(\alpha_i)(cos^2(\beta) - 1)}{cos^2(\beta)}} \quad (39)$$

To simplify equations, suppose:

$$\phi_i = \frac{sin^2(\alpha_i)(cos^2(\beta) - 1)}{cos^2(\beta)} \quad (40)$$

Regarding to 39 and 40, we conclude that:

$$S = \sum_{i=1}^{n} \sqrt{1 - \emptyset_i} \quad (41)$$

$$S^2 = (\sum_{i=1}^{n} \sqrt{1 - \emptyset_i})^2 \quad (42)$$

$$S^2 = (\sum_{i=1}^{n}(1 - \emptyset_i)) + 2((\sqrt{1-\emptyset_1}\sqrt{1-\emptyset_2}) + (\sqrt{1-\emptyset_1}\sqrt{1-\emptyset_3}) + \cdots + (\sqrt{1-\emptyset_{n-1}}\sqrt{1-\emptyset_n})) \quad (43)$$

$$P = 2\left((\sqrt{1-\emptyset_1}\sqrt{1-\emptyset_2}) + (\sqrt{1-\emptyset_1}\sqrt{1-\emptyset_3}) + \cdots + (\sqrt{1-\emptyset_{n-1}}\sqrt{1-\emptyset_n})\right) \quad (44)$$

So:

$$S^2 = n - \sum_{i=1}^{n} \emptyset_i + P \quad (45)$$

Regarding to 40 and 45, we conclude that:

$$S^2 = n - \left(\sum_{i=1}^{n} \frac{sin^2(\alpha_i)(cos^2(\beta) - 1)}{cos^2(\beta)}\right) + P \quad (46)$$

$$S^2 = n - \left(\frac{cos^2(\beta) - 1}{cos^2(\beta)} \sum_{i=1}^{n} sin^2(\alpha_i)\right) + P \quad (47)$$

We have:

$$\sum_{i=1}^{n}(sin^2(\alpha_i) + cos^2(\alpha_i)) = \sum_{i=1}^{n} sin^2(\alpha_i) + \sum_{i=1}^{n} cos^2(\alpha_i) = n \quad (48)$$

So:

$$\sum_{i=1}^{n} sin^2(\alpha_i) = \frac{n}{2} \quad (49)$$

Regarding to 47 and 49, we conclude that:

$$S^2 = n - \frac{n(cos^2(\beta) - 1)}{2cos^2(\beta)} + P \quad (50)$$

$$P = n(n-1) \times average_P = (n^2 - n) \times average_P \quad (51)$$

Regarding to 50 and 51, we conclude that:

$$S^2 = n - \frac{n(cos^2(\beta) - 1)}{2cos^2(\beta)} + n^2.average_p - n.average_p \quad (52)$$

$$S^2 = n^2\left(\frac{1}{n} - \frac{cos^2(\beta) - 1}{2ncos^2(\beta)} - \frac{average_p}{n} + average_p\right) \quad (53)$$

$$S = n\sqrt{\frac{1}{n} - \frac{cos^2(\beta) - 1}{2ncos^2(\beta)} - \frac{average_p}{n} + average_p} \quad (54)$$

Regarding to 37 and 54, we conclude that:

$$Ellipse\ Circumference = \lim_{n\to\infty}(2b.\sin(\frac{180^\circ}{4n}) \times 4n\sqrt{\frac{1}{n} - \frac{cos^2(\beta) - 1}{2ncos^2(\beta)} - \frac{average_p}{n} + average_p}) \quad (55)$$

So:

$$Ellipse\ Circumference = 2\pi b\sqrt{average_p} \quad (56)$$

Calculation of average$_p$

If $\sqrt{1-\emptyset_k}\sqrt{1-\emptyset_k}$ is the closest quantity to the avarage$_P$, then:

$average_p = \sqrt{1-\emptyset_k}\sqrt{1-\emptyset_k} = 1 - \emptyset_k$, where 1≤k≤n (57)

I have designed a java program to calculate $sin^2(\emptyset_k)_i$ and $(\frac{b}{a})_i$ and plot them on a scatter diagram (Figure 7).

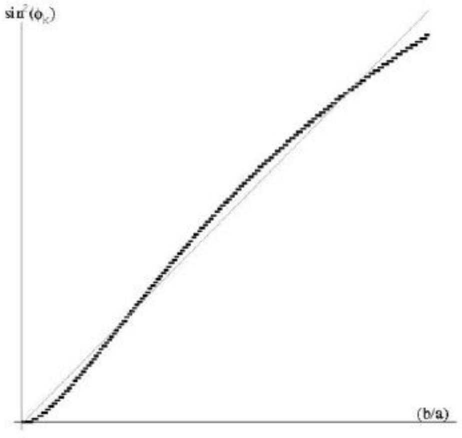

Figure 7: Scatter plot of $sin^2(\emptyset_k)_i$ and $(\frac{b}{a})_i$

The algorithm of program has been displayed in figure 8.

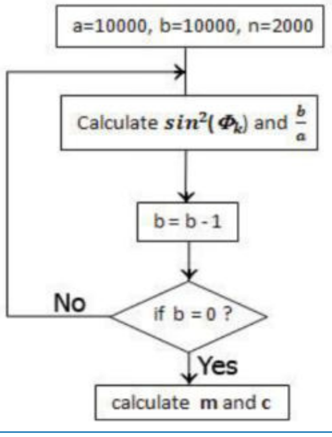

Figure 8: Algorithm of java program

I have used regression analysis to obtain a linear equation between $sin^2(\emptyset_k)$ and $\frac{b}{a}$. Regression line associated with *n* points (x_i, y_i) is equal to: [6]

$$y = mx + c \quad (58)$$

Where:

$$m = \frac{n\sum(xy) - \sum x \sum y}{n\sum x^2 - (\sum x)^2} \quad (59)$$

$$c = \frac{\sum y - m\sum x}{n} \quad (60)$$

Then:

$$m = \frac{n\left(\sum_{i=1}^{10^4} sin^2(\emptyset_k)_i \left(\frac{b}{a}\right)_i\right) - \left(\sum_{i=1}^{10^4} sin^2(\emptyset_k)_i\right)\left(\sum_{i=1}^{10^4}\left(\frac{b}{a}\right)_i\right)}{n\left(\sum_{i=1}^{10^4}\left(\left(\frac{b}{a}\right)_i\right)^2\right) - \left(\sum_{i=1}^{10^4}\left(\frac{b}{a}\right)_i\right)^2} \quad (61)$$

$$c = \frac{\sum_{i=1}^{10^4} sin^2(\emptyset_k)_i - m\sum_{i=1}^{10^4}\left(\frac{b}{a}\right)_i}{10^4} \quad (62)$$

So:

$$sin^2(\emptyset_k)_i = m\left(\frac{b}{a}\right)_i + c \quad (63)$$

Regarding to 1, 40, 57 and 63, we conclude that:

$$average_p = 1 - \emptyset_k = 1 - \frac{\left(m\frac{b}{a} + c\right)\left(\frac{b^2}{a^2} - 1\right)}{\frac{b^2}{a^2}} \quad (64)$$

$$average_p = 1 - \left(m\frac{b}{a} + c\right)\left(\frac{b^2}{a^2} - 1\right)\left(\frac{a^2}{b^2}\right) \quad (65)$$

$$average_p = 1 - \left(1 - \frac{a^2}{b^2}\right)\left(m\frac{b}{a} + c\right) \quad (66)$$

Formula of ellipse circumference

Finally, regarding to 56 and 66, we conclude:

$$Ellipse\ Circumference = 2\pi b\sqrt{1 - \left(1 - \frac{a^2}{b^2}\right)\left(m\frac{b}{a} + c\right)} \quad (67)$$

and

$$m = 0.40521399808021946 \ and \ c = 0.10071256100281531$$

Where: $a = 10000, b = 10000 \ and \ n = 2000$ $(to\ obtain\ regression\ line)$.

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