ORIGINAL RESEARCH PAPER

Mathematics

A STUDY ON LINEAR TRANSFORMATION AND ITS PROPERTIES

KEY WORDS: Field, Binary operations, Vector spaces, Subspaces, Linear transformation.

Anil.S.C*	Assistant professor, JNNCE , Shivamogga*Corresponding Author
Shaila.S.Bhat	Assistant professor, JNNCE , Shivamogga
Bhagyala kshmi.k	Assistant professor, JNNCE , Shivamogga
Dr.Gurupadavva Ingalahalli	Assistant professor, JNNCE , Shivamogga

ABSTRACT

Linear algebra is the branch of algebra in which we study vector spaces, linear dependence and independence, dimension, subspaces, and linear transformations. A linear transformation between two vector spaces is a rule that assigns a vector in one space to a vector in the other space. This paper is a study of the properties of a linear transformation.

Introduction:
Let F be a field and V be a non empty set defined under two binary operations addition and scalar multiplication, then V is said to ba a vector space if the following axioms are satisfied.

1. V is an abelian group under addition, ie

$\alpha, \beta \in V, \alpha + \beta \in V$
$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)\ \forall\, \alpha, \beta, \gamma \in V$
$\exists 0 \in V, \ni \alpha + 0 = 0 + \alpha = \alpha$
$\forall \alpha \in V, \exists - \alpha \in V$ such that
$\alpha + (-\alpha) = (-\alpha) + \alpha = 0$
$\alpha + \beta = \beta + \alpha, \forall \alpha, \beta \in V$

2. a∈F, ∈V implies a ∈V, ∀a∈F

$a(\alpha + \beta) = a\alpha + a\beta, \forall a \in F$ and $\alpha, \beta \in V$
$(a + b)\alpha = a\alpha + b\alpha, \forall a, b \in F$ and $\alpha \in V$
$(ab)\,\alpha = a(b\alpha) \forall a, b \in F$ and $\alpha \in V$
$\exists 1 \in F$ such that $1.\alpha = \alpha.1 = \alpha, \forall \alpha \in V$

Definition:
A non-empty sub set w of a vector space V over a field F is called a subspace of V if w itself is a vector space over F under the same operations of addition and scalar multiplication as defined in V Zero space and vector space V are subspaces of V over F and are called improper subspaces or trivial subspaces of V, all other subspaces of V are called proper subspaces of V.

Theorem: A non-empty subset w of a vector space V over a field F is a sub space of V if

i) $\forall \alpha, \beta \in w$, then $\alpha + \beta \in w$

ii) $c \in F$ and $\alpha \in w$, then $c.\alpha \in w$

Theorem: A non-empty sub set w is a sub space of a vector space V over F iff
$a\alpha + b\beta \in w\ \forall \alpha, \beta \in w$ and $a, b \in F$

Linear transformations:
Let U and V be two vector spaces over the same field F, the mapping T:U V is said to be a linear transformation if,

$T(\alpha + \beta) = T(\alpha) + T(\beta), \forall \alpha, \beta \in U$

T(c $T(c\alpha) = cT(\alpha), \forall c \in F, \forall \alpha \in U$

Linear transformation is also called as linear map or linear operator.

Theorem : A mapping T:U V , from a vector space U(F) in to V(F) is a linear transformation iff

$T(c_1\alpha + c_2\beta) = c_1T(\alpha) + c_2T(\beta), \ \forall c_1, c_2 \in F$ and $\forall \alpha, \beta \in U$

Proof: Suppose T:U V is a linear transformation , then

$$T(c_1\alpha + c_2\beta) = T(c_1\alpha) + T(c_2\beta) = c_1T(\alpha) + c_2T(\beta)$$
$\forall c_1, c_2 \in F, \alpha, \beta \in U$

Conversely,

$T(c_1\alpha + c_2\beta) = c_1T(\alpha) + c_2T(\beta)$
$\forall c_1, c_2 \in F, \alpha, \beta \in U$
In particular, $c_1 = 1, c_2 = 1$,
we get $T(\alpha + \beta) = T(\alpha) + T(\beta)$,
Again , let $c_2 = 0$, we get
$T(c_1\,\alpha) = c_1\,T(\alpha)$,
⇨ T is a linear transformation.

Properties of linear transformation:
THEOREM: If T:U V linear transformation then

a) $T(0) = 0'$, where 0 and $0'$ are zero vectors of U and V respectively

b) $T(-\alpha) = -T(\alpha), \forall \alpha \in U$

c) $T(c_1\alpha_1 + c_2\alpha_2 + \cdots \ldots \ldots \ldots . + c_n\alpha_n) = c_1T(\alpha_1) + c_2T(\alpha_2) + \cdots \ldots \ldots \ldots + c_nT(\alpha_n)$

d) $T(\alpha - \beta) = T(\alpha) - T(\beta)$

Proof:
a) Let $\alpha \in U$

$T(\alpha + 0) = T(\alpha) + T(0)$

⇨ $T(\alpha) = T(\alpha) + T(0)$

⇨ $T(\alpha) + 0' = T(\alpha) + T(0)$

⇨ $0' = T(0)$, by left cancellation law

b) consider $T[\alpha+(-\alpha)]=T(\alpha)+T(-\alpha)$

$T(0) = T(\alpha) + T(-\alpha)$

$0' = T(\alpha) + T(-\alpha)$

$T(\alpha)$ is additive inverse of $T(-\alpha)$

$T(-\alpha) = -T(\alpha)$

c) We shall prove the result by mathematic induction

Let $P(n): T(c_1\alpha_1 + c_2\alpha_2 + \cdots \dots \dots + c_n\alpha_n) = c_1T(\alpha_1) + c_2T(\alpha_2) + \cdots \dots + c_nT(\alpha_n)$

Now $P(1): T(c_1\alpha_1) = c_1T(\alpha_1)$[T is linear]

⇨ P(1) is true

Let us assume that the result is true for some positive integer k

$P(k): T(c_1\alpha_1 + c_2\alpha_2 + \cdots \dots \dots \dots + c_k\alpha_k) = c_1T(\alpha_1) + c_2T(\alpha_2) + \cdots \dots + c_kT(\alpha_k)$

Now we prove that the result is true for $n = k + 1$

$T(c_1\alpha_1 + c_2\alpha_2 + \cdots . + c_k\alpha_k + c_{k+1}\alpha_{k+1})$

$= T(c_1\alpha_1 + c_2\alpha_2 + \dots . + c_k\alpha_k) + T(c_{k+1}\alpha_{k+1})$

$= c_1T(\alpha_1) + c_2T(\alpha_2) + \cdots \dots \dots \dots + c_kT(\alpha_k) + c_{k+1}T(\alpha_{k+1})$

⇨ P(k + 1) is true

⇨ thus by induction P(n) is true for all integer

d) $T(\alpha - \beta) = T(\alpha) + T(-\beta) = T(\alpha) - T(\beta)$

Theorem: If $\beta_1, \beta_2 \dots \dots \dots \dots \dots \beta_n$ be any basis of a vector space V and $a_1, a_2, \dots \dots \dots a_m$ be any m vectors of the vector space w, then there exists one and only one linear transformation

$T: V \to w$ with $T(\beta_i) = \alpha_i$

for $i = 1,2,3 \dots \dots \dots m$

Proof: Let $\alpha \in V$,

⇨ $\alpha = c_1\beta_1 + c_2\beta_2 \dots + c_m\beta_m$
$[\beta_1, \beta_2 \dots \dots \dots \dots \dots . \beta_m$ basis of v]

Define $T: V \to w$ by
$T(\alpha) = c_1\alpha_1 + c_2\alpha_2 + \cdots .. c_m\alpha_m$

ie $T(c_1\beta_1 + c_2\beta_2 + \cdots \dots \dots \dots \dots \dots + c_m\beta_m)$

$= c_1\alpha_1 + c_2\alpha_2 + \cdots \dots \dots \dots .. c_m\alpha_m$

We prove that
a) T is linear
b) $T(\beta_i) = \alpha_i$
c) T is unique

a) Let $\alpha, \beta \in V$

$\alpha = c_1\beta_1 + c_2\beta_2 + \cdots \dots \dots \dots \dots + c_m\beta_m$,

$\beta = d_1\beta_1 + d_2\beta_2 + \cdots \dots \dots \dots \dots + d_m\beta_m$

$T(\alpha + \beta) = T[(c_1 + d_1)\beta_1 + (c_2 + d_2)\beta_2 + \cdots \dots \dots \dots \dots + (c_m + d_m)\beta_m]$

$= (c_1 + d_1)\alpha_1 + (c_2 + d_2)\alpha_2 + \cdots \dots \dots \dots \dots \dots + (c_m + d_m)\alpha_m$

$= c_1\alpha_1 + c_2\alpha_2 + \cdots \dots .. c_m\alpha_m + d_1\beta_1 + d_2\beta_2 + \cdots \dots \dots + d_m\beta_m$

$= T(\alpha) + T(\beta)$

Let $c \in F$,
$T(c\alpha) = T(cc_1\beta_1 + cc_2\beta_2 + \cdots \dots \dots \dots + cc_m\beta_m)$
$= c(c_1\alpha_1 + c_2\alpha_2 + \cdots \dots \dots \dots .. c_m\alpha_m)$
$= cT(\alpha)$

b) Let $\beta_i \in V$ and

$\beta_i = 0\beta_1 + 0\beta_2 + \cdots \dots + 1\beta_i + \cdots \dots \dots . + 0\beta_m$

$T(\beta i) = 0\alpha_1 + 0\alpha_2 + \cdots \dots \dots . + 1\alpha_i + \cdots \dots . . + 0\alpha_m$

⇨ $T(\beta_i) = \alpha_i, i = 1,2,3,4 \dots \dots \dots \dots .$

c) To show that T is unique

Let F be another linear transformation such

that $F(\beta i) = \alpha i$, $i = 1,2,3 \dots \dots ..$

$F(\alpha) = F(c_1\beta_1 + c_2\beta_2 + \cdots \dots \dots \dots \dots + c_m\beta_m)$

$= F(c_1\beta_1) + F(c_2\beta_2) + \cdots \dots . + F(c_m\beta_m)$

$= c_1F(\beta_1) + c_2F(\beta_2) + \cdots .. + c_mF(\beta_m)$

⇨ $F(\alpha) = c_1\alpha_1 + c_2\alpha_2 + \cdots \dots \dots \dots .. c_m\alpha_m$

⇨ $F(\alpha) = T(\alpha)$

⇨ T is unique

Conclusions:

Linear transformations are used in both abstract mathematics, as well as computer science.The additive property of the linear transformation is that the out put will be the same if the numbers are added first and then transformed or if they are transformed and then added together ie $T(a+\beta)=T(a)+T(\beta)$. The scalar multiplication of the linear transformation is that the out put will be the same if the variable a is multiplied by scalar and then transformed or if it is transformed and then multiplied ie $T(ca)=cT(a)$

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